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Alexandra Diana MELEȘTEU

DOCTORAL THESIS

Approximation methods using linear and positive operators

SUMMARY

Conducător ştiințific

Prof.dr. Radu PĂLTĂNEA

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Doctoral Advisory Committee:

Conf. dr. Ion-Gabriel Stan	Chair of committee, Transilvania University of Brașov
Prof.dr. Radu Păltănea	Doctoral supervisor, Transilvania University of Brașov
Prof.dr. Octavian Agratini	Member of committee, Tiberiu Popoviciu Institute of Numerical Analysis, Romanian Academy, Cluj Napoca
Prof.dr. Ana-Maria Acu	Member of committee, "Lucian Blaga" University of Sibiu
Conf. dr. Nicușor Minculete	Member of committee, Transilvania University of Brașov

Contents

The main topic

The present work is dedicated to methods of approximation using linear positive operators. These operators are essential in the theory of function approximation, as they not only provide arbitrarily good approximations of functions, but also have the capacity to preserve certain special properties of the functions.

A fundamental result in the theory of continuous function approximation is the possibility of arbitrarily good approximation in the uniform norm of a continuous function on a closed and bounded interval by algebraic polynomials. This was demonstrated by the German mathematician Karl Weierstrass at the end of the 19th century. Additionally, this theorem can be proved using sequences of polynomial operators constructed with linear positive operators, such as the sequence of Bernstein polynomials.

The theory of approximation by linear positive operators, particularly developed after the independent discoveries by mathematicians Popoviciu, Bohman, and Korovkin, who established simple conditions for uniform approximation of a continuous function using such operators.

In recent decades, Romanian mathematicians have also made significant contributions in this field.

Basic tools for evaluating the order of approximation of functions include moduli of continuity and K-functionals.

The theory of approximation by linear positive operators employs the application of general methods from classical analysis, functional analysis, probability theory, etc.

The presented work contains several chapters addressing different aspects of approximation theory. This reflects the variety of problems and methods that arise in approximation theory. The common foundation of these studies is linear and positive operators.

For the realization of this work, I benefited from the guidance received from PhD supervisor: Professor Radu Păltănea, with whom I have also published a paper, and the members of the supervisory committee: Professor Marin Marin, Professor Dorina Răducanu, Associate Professor Marius Birou, Professor Mihai N. Pascu, and Lecturer Maria Talpău Dimitriu, with the last two of them having also works written in collaboration. I am deeply grateful to all of them.

List of notations, symbols, and abbreviations

We will introduce the notations necessary for a complete understanding of this work.

i) $F(X)$ denotes the space of all real functions defined on the set X;
$C(X)$ denotes the space of all real continuous functions on the set X;
$B(X)$ denotes the space of all real bounded functions on the set X;
$C_B(X) := C(X) \cap B(X)$;
$W^2_\infty([a,b])$ denotes the space of all functions f on the interval $[a,b]$ for which f' is absolutely continuous and $|f''| \leq M$ for some M;
$UC_B(X)$ denotes the space of all real bounded and uniformly continuous functions on the set X;

ii) $L_p(I)$ denotes the space of classes of functions defined on I with integrable p-th power of the modulus, $p > 0$;

iii) $L(f)$ or Lf denotes the function obtained by applying the operator L to the function f, and $L(f)(x)$ denotes the value of $L(f)$ evaluated at the point x. Occasionally, we also denote this as $(Lf)(x)$ and $L(f,x)$ instead of $L(f)(x)$;

iv) $e_j(t) = t^j, j = 0,1,2,...$ denotes the monomial functions;

v) $\Psi(x) = x(1-x)$ denotes a function defined on the interval $[0,1]$;

vi) $B(x,y)$ denotes the Beta function with two variables $x > 0$ and $y > 0$, defined as: $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ and $\Gamma(x)$ denotes the Gamma function, defined as: $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$;

vii) We use standard notations:

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad \mathbb{R}^s_+ = \{\overline{x} = (x_1, ..., x_s) \mid x_i \geq 0\},$$

$$\overline{x} = (x_1, ..., x_s) \in \mathbb{R}^s_+, \ s \in \mathbb{N}, \quad |\overline{x}| = x_1 + ... + x_s,$$

$$\overline{k} = (k_1, ..., k_s) \in \mathbb{N}^s_0, \quad |\overline{k}| = k_1 + ... + k_s, \quad \overline{k}! = (k_1!, ..., k_s!),$$

$$\overline{x}^{\overline{k}} = (x_1^{k_1}, ..., x_s^{k_s}), \quad \binom{n}{\overline{k}} = \frac{n!}{\overline{k}! \cdot (n - |\overline{k}|)!}, \ n \in \mathbb{N}, \quad \frac{\overline{k}}{n} = \left(\frac{k_1}{n}, ..., \frac{k_s}{n}\right),$$

$$\Delta = \{\overline{x} \in \mathbb{R}^s_+ \mid |\overline{x}| \leq 1\}, \quad \Lambda_n = \{\overline{k} \in \mathbb{N}^s_0 \mid |\overline{k}| \leq n\}. \tag{1}$$

Chapter 1

Introduction

In this chapter, we will highlight the theoretical foundations underlying the studied domain.

1.1 General theorems of function approximation by linear and positive operators

The theorem formulated by the Romanian mathematician Tiberiu Popoviciu ([56]) is the first general theorem providing sufficient conditions, expressed using the second-order moment, for certain sequences of linear and positive operators to uniformly approximate continuous functions.

Bochmann's theorem ([14]) shows that for a sequence of operators from a certain class of linear and positive operators to have approximation properties, it is sufficient for the sequence to successfully approximate three functions, called test functions, which are formed from the first monomial functions.

Korovkin's theorem ([34]) extends Bochmann's theorem by using general sequences of linear positive operators and as test functions any three functions that form a Chebyshev system [7].

In the following, we will present the three theorems mentioned above.

Theorem 1.1.1. *(Popoviciu, [56])*

Let $L_n : C([a,b]) \longrightarrow C([a,b])$ *be a sequence of operators of the form:*

$$L_n(f)(x) = \sum_{i=1}^{m_n} f(x_{n,i})\rho_{n,i}(x),$$

with $x_{n,i} \in [a,b]$*,* $\rho_{n,i}(x) \geq 0$*,* $x \in [a,b]$*,* $m_n \in \mathbb{N}$*.*

If:

i) $L_n(e_0)(x) = 1$*,* $x \in [a,b]$*;*

ii) $\lim\limits_{n\longrightarrow\infty} L_n((e_1 - x)^2)(x) = 0$ *uniformly with respect to* x*, then*

$$\lim_{n\longrightarrow\infty} L_n(f)(x) = f(x)$$

uniformly on $[a,b]$*, for all* $f \in [a,b]$*.*

Theorem 1.1.2. *(Bochmann, [14])*
Let $L_n : C([a,b]) \longrightarrow C([a,b])$ *be a sequence of operators of the form:*

$$L_n(f)(x) = \sum_{i=1}^{n} f(x_{n,i})\rho_{n,i}(x),$$

with $f \in C([a,b])$*,* $x_{n,i} \in [a,b]$*,* $\rho_{n,i}(x) \geq 0$*.*
If:

$$\lim_{n\longrightarrow\infty} L_n(e_j)(x) = e_j(x),\ j = 0,1,2$$

uniformly with respect to $x \in [a,b]$*, then:*

$$\lim_{n\longrightarrow\infty} L_n(f)(x) = f(x)$$

uniformly for $x \in [a,b]$*, for all* $f \in [a,b]$*.*

Definition 1.1.1. *An operator* $L : C([a,b]) \longrightarrow F([a,b])$ *is called linear and positive if it satisfies the conditions:*

i) Linearity: $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$*, for* $f, g \in C([a,b])$*,* $\alpha, \beta \in \mathbb{R}$*;*

ii) Positivity: $f \geq 0 \Rightarrow L(f) \geq 0$*.*

Theorem 1.1.3. *(Korovkin, [34])*
Let $L_n : V \subset C([a,b]) \longrightarrow F([a,b])$ *be a sequence of linear and positive operators, where* V *is a subspace of* $F([a,b])$ *containing the functions* $\rho_0, \rho_1, \rho_2 \in C([a,b])$ *that form a Chebyshev system.*
If:

$$\lim_{n\longrightarrow\infty} L_n(\rho_j)(x) = \rho_j(x),\ j = 0,1,2$$

uniformly with respect to $x \in [a,b]$*,* $j = 0,1,2,...$*, then:*

$$\lim_{n\longrightarrow\infty} L_n(f)(x) = f(x)$$

uniformly for $x \in [a,b]$*, for all* $f \in [a,b]$*.*

The periodic variant of this theorem is as follows.

Theorem 1.1.4. *(Korovkin, [34])*
Let the space be

$$C_{2\pi}(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid f \text{ is } 2\pi\text{-periodic}\}.$$

Let $(L_n)_{n\geq 1}$ *be a sequence of linear and positive operators from* $C_{2\pi}(\mathbb{R})$ *to* $F(\mathbb{R})$ *such that:*

$$\lim_{n\to\infty} L_n(g) = g$$

uniformly on $\mathbb{R}$ *for any* $g \in \{1, \cos, \sin\}$.

Then:

$$\lim_{n\to\infty} L_n(f) = f$$

uniformly on $\mathbb{R}$ *for any* $f \in C_{2\pi}(\mathbb{R})$.

It is important to note that Korovkin's theorem cannot be applied to continuous functions defined on an unbounded interval. There are results that extend Korovkin's theorem by imposing additional conditions. We mention the following results.

Theorem 1.1.5. *(Altomare, [7])*

Let (X, d) *be a metric space and* E *a lattice subspace of* $F(X)$ *containing all constant functions and all functions* d_x^2*, where we denote* $d_x(y) = d(x, y)$. *Let* $(L_n)_{n\geq 1}$ *be a sequence of linear and positive operators from* E *to* $F(X)$*, and* Y *a subset of* X *such that:*

i) $\lim_{n\to\infty} L_n(1) = 1$ *uniformly on* Y*;*

ii) $\lim_{n\to\infty} L_n(d_x^2) = 0$ *uniformly with respect to* $x \in Y$.

Then for all $f \in E \cap UC_B(X)$*, it holds that* $\lim_{n\to\infty} L_n(f) = f$ *uniformly on* Y.

Theorem 1.1.6. *(Altomare, [7])*

Let (X, d) *be a locally compact metric space and* E *a lattice subspace of* $F(X)$ *containing all constant functions* 1 *and all functions* d_x^2. *Let* $(L_n)_{n\geq 1}$ *be a sequence of linear and positive operators from* E *to* $F(X)$ *such that:*

i) $\lim_{n\to\infty} L_n(1) = 1$ *uniformly on* X*;*

ii) $\lim_{n\to\infty} L_n(d_x^2) = 0$ *uniformly on compact subsets of* X.

Then for all $f \in E \cap UC_B(X)$*, it holds that* $\lim_{n\to\infty} L_n(f) = f$ *uniformly on compact subsets of* X.

The general form of a Voronovskaja-type theorem for a sequence of linear and positive operators $(L_n)_n$ is as follows:

$$\lim_{n\to\infty} \alpha_n(L_n(f)(x) - f(x)) = E(x, f'(x), f''(x), \ldots). \tag{1.1}$$

1.2 Moduli of continuity and K-functionals

Estimation using moduli of continuity is a commonly used technique in approximation theory. The modulus of continuity measures the variation of a function's values over an interval or, in other words, the uniform continuity of a function over an interval.

K-functionals represent an important tool in approximation theory and are equivalent to moduli of continuity. They provide information about functions using their approximation by smoother functions.

Definition 1.2.1. *Let $f \in B([a,b])$ and $h > 0$.*

Define the modulus of continuity of first order as:

$$\omega(f,h) = \sup\{|f(x) - f(y)| \mid x, y \in [a,b], |x-y| \le h\}. \tag{1.2}$$

Proposition 1.2.1. *(DeVore, Lorenz, [23])*

The properties of the modulus ω are as follows:

i) $\omega(f+g,h) \le \omega(f,h) + \omega(g,h), \quad f,g \in B([a,b]), h > 0;$

ii) $\omega(f,h_1+h_2) \le \omega(f,h_1) + \omega(f,h_2), \quad f \in B([a,b]), h_1, h_2 > 0;$

iii) $f \in C([a,b]) \Rightarrow \lim\limits_{h\to 0} \omega(f,h) = 0, \quad h > 0;$

iv) $\omega(f,\lambda) \le (1 + \frac{\lambda}{\delta})\omega(f,\delta), \quad f \in B([a,b]), h > 0, \lambda > 0, \delta > 0;$

v) $\omega(f,h_1) \le \omega(f,h_2), \quad f \in B([a,b]), 0 \le h_1 \le h_2.$

Definition 1.2.2. *Let $f \in B([a,b])$ and $h > 0$.*

Define the modulus of continuity of k th order as:

$$\omega_k(f,h) = \sup\{|\Delta_\rho^k f(x)| \mid x, x+k\rho \in [a,b], 0 < \rho \le h\}, \tag{1.3}$$

where

$$\Delta_\rho^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+k\rho). \tag{1.4}$$

Proposition 1.2.2. *The properties of the modulus ω_k are as follows:*

i) $\omega_k(f+g,h) \le \omega_k(f,h) + \omega_k(g,h), \quad f,g \in B([a,b]), h > 0;$

ii) $f \in C([a,b]) \Rightarrow \lim\limits_{h\to 0} \omega_k(f,h) = 0, \quad h > 0;$

iii) $\omega_k(f,\lambda) \le \left[1 + \left(\frac{\lambda}{\delta}\right)^k\right] \omega_k(f,\delta), \quad f \in B([a,b]), h > 0, \lambda > 0, \delta > 0;$

iv) $\omega_k(f,h_1) \le \omega_k(f,h_2), \quad f \in B([a,b]), 0 \le h_1 \le h_2.$

Theorem 1.2.1. *(Shisha and Mond, [60])*

Let $L_n : C([a,b]) \to F([a,b])$ *be a sequence of linear and positive operators. We have:*

$$|L_n(f)(x) - f(x)| \leq |f(x)| \cdot |L_n(e_0)(x) - 1|$$
$$+ \left(L_n(e_0)(x) + \frac{1}{h}\sqrt{L_n((e_1 - x)^2)(x) L_n(e_0)(x)} \right) \omega_1(f,h),$$

where $f \in C([a,b]), x \in [a,b],\ h > 0.$

Remark 1.2.1. If we take: $h = \sqrt{L_n((e_1 - x)^2)(x) L_n(e_0)(x)}$, we get:

$$|L_n(f)(x) - f(x)| \leq |f(x)| \cdot |L_n(e_0)(x) - 1| + (1 + L_n(e_0)(x))\, \omega_1(f, \mu_n(x)),$$

where:

$$\mu_n(x) = \sqrt{L_n((e_1 - x)^2)(x) L_n(e_0)(x)}.$$

Remark 1.2.2. If in the previous theorem we choose $h = \mu_n$, where μ_n is the norm of the function $\mu_n(x)$, we obtain:

$$\| L_n(f) - f \| \leq \| f \| \| L_n(e_0) - e_0 \| + (1 + \| L_n(e_0) \|) \omega_1(f, \mu_n).$$

From this formula, it follows that if the sequence of operators L_n satisfies the conditions of Korovkin's theorem, then $L_n f$ converges uniformly to f; this is because in this case, $\mu_n \to 0$.

Theorem 1.2.2. *(Păltănea, [53])*

If $L : C([a,b]) \to C([a,b])$ *is a linear and positive operator, with the property that* $L(e_j) = e_j, j = 0, 1,$ *then:*

$$\| L(f) - f \| \leq \frac{3}{2} \omega_2(f, \mu),$$

where $\mu = \sup\limits_{x \in [a,b]} \sqrt{L((e_i - x)^2, x)}.$

Definition 1.2.3. *Let* $(X, \| \cdot \|_X)$ *be a normed space. Let* $Y \subset X$ *be a subspace endowed with a seminorm* $| \cdot |_Y$*. Define:*

$$K(X, Y, x, t) = \inf\{\| x - y \| + t \mid y \in Y\},\ x \in X,\ t > 0.$$

Definition 1.2.4. *In the particular case where* $X = C([a,b])$ *and* $Y = C^k([a,b])$*, with* $k \in \mathbb{N}$ *and* $k \geq 1$*, we have:*

$$K_k(f, t^k) = K(C([a,b]), C^k([a,b]), f, t^k) = \inf\{\| f - g \| + t^k \| g^{(k)} \| | g \in C^k([a,b])\}.$$

Theorem 1.2.3. *(Johnen, [32])*

For any interval $[a,b]$ and any $k \in \mathbb{N}$, there exist constants C_1^k and C_2^k depending only on k and the interval $[a,b]$, such that:

$$C_1^k \omega_k(f,t) \leq K_k(f,t^k) \leq C_2^k \omega_k(f,t), \quad \forall f \in C([a,b]), \forall t > 0.$$

We now present the classical types of operators that we will use in this work.

1.3 Bernstein operators

Bernstein operators represent a class of operators that have captivated mathematicians' attention due to their elegant and efficient ability to approximate functions. Their name is derived from the Russian mathematician Sergey Natanovich Bernstein, who played a crucial role in the development of approximation theory and in formulating significant results associated with these operators.

Besides their ability to approximate continuous functions, Bernstein operators have a number of other remarkable properties, such as simultaneous approximation and preservation of certain classes of functions. Furthermore, they are easy to compute, which makes them particularly useful in practice.

These operators have served as a starting point for the construction of many other approximation operator sequences. There is a extensive literature related to Bernstein operators, including notable works such as the monograph [37] and the recent monograph [17].

Essentially, Bernstein operators are constructed using fundamental polynomials, denoted by $b_{n,k}$, where $0 \leq k \leq n$, defined below, which form an algebraic basis for polynomials of degree n, known as the Bernstein basis.

Let $n \in \mathbb{N}$. We define the Bernstein operators $B_n : C([0,1]) \to C([0,1])$ by:

$$B_n(f)(x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = \overline{0,n},\ x \in [0,1].$$

Theorem 1.3.1. *(Bustamante, [17])*

The Bernstein operators have the following properties:

i) *They are linear and positive;*

ii) *For any $n \geq 1$, we have:*

$$B_n(f)(0) = f(0) \quad \text{and} \quad B_n(f)(1) = f(1);$$

iii)

$$\sum_{k=0}^{n} b_{n,k}(x) = 1;$$

iv)
$$B_n(e_0)(x) = 1; \quad B_n(e_1)(x) = x; \quad B_n(e_2)(x) = x^2 + \frac{x(1-x)}{n};$$

v)
$$\|B_n\| = 1;$$

vi) *(1) If f is convex on $[0,1]$, then*
$$\forall n \in \mathbb{N}, \forall x \in (0,1), \quad B_n(f)(x) \geq B_{n+1}(f)(x),$$
and if f is concave on $[0,1]$, then the inequality is reversed;

(2) If f is convex on $[0,1]$, then
$$\forall n \in \mathbb{N}, \forall x \in (0,1), \quad B_n(f)(x) \geq f(x),$$
and if f is concave on $[0,1]$, then the inequality is reversed;

vii) There exists a constant $C > 0$ such that
$$|B_n(f)(x) - f(x)| \leq C\omega(f, \frac{1}{\sqrt{n}}), \quad \forall x \in [0,1];$$

viii) There exists a constant $C > 0$ such that
$$|B_n(f)(x) - f(x)| \leq C\frac{1}{\sqrt{n}}\omega(f', \frac{1}{\sqrt{n}}), \quad \forall f \in C^1([0,1]), \quad \forall x \in [0,1].$$

Remark 1.3.1. The integral of the Bernstein polynomial $b_{n,\nu}(x)$ over the interval $[0,1]$ can be computed using the Beta function:
$$\begin{aligned}\int_0^1 b_{n,\nu}(x)\,dx &= \binom{n}{\nu}\int_0^1 x^\nu(1-x)^{n-\nu}\,dx \\ &= \frac{1}{n+1}.\end{aligned}$$

Next, we present some fundamental results regarding Bernstein operators.

Theorem 1.3.2. *(Lorentz, [38])*
For a bounded function $f(x)$ on $[0,1]$, the relation
$$\lim_{n\to\infty} B_n(f)(x) = f(x)$$
holds at every point of continuity x of the function f. This relation also holds uniformly over the interval $[0,1]$ if $f(x)$ is continuous on the interval.

Recall the notation $\Psi(x) = x(1-x)$, $x \in [0,1]$.

Theorem 1.3.3. *(Voronovskaja, [72])*
If f is bounded on $[0,1]$, differentiable in a neighborhood of x, and has a second derivative $f''(x)$ for $x \in [0,1]$, then:

$$\lim_{n\to\infty} n\left[B_n(f)(x) - f(x)\right] = \frac{\Psi(x)}{2} f''(x). \tag{1.5}$$

If $f \in C^2([0,1])$, the convergence is uniform.

Theorem 1.3.4. *(Păltănea, [53])*
For any function $f \in C([0,1])$, we have:

$$|f(x) - B_n(f)(x)| \leq \frac{3}{2}\omega_2\left(f, \sqrt{\frac{\Psi(x)}{n}}\right), \quad x \in [0,1].$$

In the particular case where $f \in C^1([0,1])$, we have:

$$|f(x) - B_n(f)(x)| \leq \frac{1}{2}\sqrt{\frac{\Psi(x)}{n}}\,\omega\left(f', 2\sqrt{\frac{\Psi(x)}{n}}\right), \quad x \in [0,1].$$

We also have the following local saturation property.

Theorem 1.3.5. *(Lorentz, [37])*
Let $M > 0$. The inequality

$$B_n(f)(x) \leq M\frac{\Psi(x)}{2n} + o_x\left(\frac{1}{n}\right), \quad x \in [0,1],\ n = 1, 2, \ldots \tag{1.6}$$

is equivalent to the condition

$$f \in W^2_\infty \text{ and } \|f''\|_\infty \leq M, \tag{1.7}$$

where $o_x\left(\frac{1}{n}\right)$ is Landau's symbol.

Considering Bernstein polynomials for complex values z outside the segment $[0,1]$, assuming $f(z)$ is analytic on a domain containing the segment $[0,1]$, we have the following result.

Theorem 1.3.6. *(Lorentz, [38])*
Let $a \in [0,1]$ such that $R \geq a$ and $R \geq 1-a$, ensuring that the segment $[0,1]$ is contained in the circle $|z-a| \leq R$. If the function:

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$$

is analytic on an open set containing the disc $|z-a| \leq R$, then:

$$\lim_{n\to\infty} B_n(z) = f(z) \text{ uniformly for } z \text{ in the disc } |z-a| \leq R.$$

1.4 Stancu operators

Stancu operators are an extension and generalization of Bernstein operators, enhancing their function approximation capabilities. Named after the Romanian mathematician D. D. Stancu [62], who introduced and significantly contributed to their development and study, these operators offer improved approximation for certain classes of functions.

While the theoretical foundation remains rooted in Bernstein polynomials, Stancu operators allow for more flexible approximation by incorporating three adjustable parameters.

Let $0 \leq \alpha \leq \beta$ and $m \in \mathbb{N}$. We define the Stancu operators as follows:

$$P_m^{(\alpha,\beta)} : C\left(\left[\frac{\alpha}{m+\beta}, \frac{m+\alpha}{m+\beta}\right]\right) \to C([0,1]),$$

$$P_m^{(\alpha,\beta)}(f)(x) = \sum_{k=0}^{m} b_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right),$$

where $b_{m,k}(x)$ are the Bernstein polynomials defined by:

$$b_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}.$$

Theorem 1.4.1. *(Stancu, [62])*

The Stancu operators have the following properties:

i) $P_m^{(\alpha,\beta)}(e_0)(x) = 1.$

ii)

$$P_m^{(\alpha,\beta)}(e_1)(x) = x + \frac{\alpha - \beta x}{m+\beta}.$$

iii)

$$P_m^{(\alpha,\beta)}(e_2)(x) = x^2 + \frac{mx(1-x) + (\alpha+\beta x)(2mx + \beta x + \alpha)}{(m+\beta)^2}.$$

iv) For any $f \in C([0,1])$*, the Stancu operators satisfy:*

$$\lim_{m\to\infty} P_m^{(\alpha,\beta)}(f)(x) = f(x) \text{ uniformly on } [0,1].$$

v) If $\alpha \in [0, \frac{1}{4}]$ *and* $\beta \in [\alpha, 2\alpha]$ *or* $\alpha \in [\frac{1}{4}, \frac{1}{2}]$ *and* $\beta \in [4\alpha^2, 2\alpha]$*, then:*

$$\left| P_m^{(\alpha,\beta)}(f)(x) - f(x) \right| \leq \frac{3}{2} \omega\left(f, \frac{1}{\sqrt{m+4\alpha^2}}\right).$$

vi) If $\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right]$ *and* $\beta \in [\alpha, 4\alpha^2]$ *or* $\alpha \geq \frac{1}{2}$ *and* $\beta \in [\alpha, 2\alpha]$*, then:*

$$\left|P_m^{(\alpha,\beta)}(f)(x) - f(x)\right| \leq \left(1 + \frac{4\alpha^2 + 1}{2(\beta + 1)}\right) \omega\left(f, \frac{1}{\sqrt{m + 4\alpha^2}}\right).$$

vii) If $\beta \in \left[\frac{1}{4}, 1\right]$ *and* $\alpha \in \left[0, \beta - \frac{\sqrt{\beta}}{2}\right]$ *or* $\beta \geq 1$ *and* $\alpha \in \left[0, \frac{\beta}{2}\right]$*, then:*

$$\left|P_m^{(\alpha,\beta)}(f)(x) - f(x)\right| \leq \left(1 + \frac{4(\beta - \alpha)^2 + 1}{2(\beta + 1)}\right) \omega\left(f, \frac{1}{\sqrt{m + 4(\beta - \alpha)^2}}\right).$$

viii) If $\beta \leq 1$ *and* $\alpha \in \left[\beta - \frac{\sqrt{\beta}}{2}, \frac{\beta}{2}\right] \cap [0, \infty]$*, then:*

$$\left|P_m^{(\alpha,\beta)}(f)(x) - f(x)\right| \leq \frac{3}{2}\omega\left(f, \frac{1}{\sqrt{m + 4(\beta - \alpha)^2}}\right).$$

In [64], D. D. Stancu introduced a new generalization:

$$(S_{n,r,s}f)(x) = \sum_{j=0}^{n-rs} p_{n-rs,j}(x) \sum_{i=0}^{s} p_{s,i}(x) f\left(\frac{j + ir}{n}\right), \tag{1.8}$$

where $f \in C([0,1])$ and $x \in [0,1]$. Here, $n \in \mathbb{N}$ and $r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ are fixed such that $rs < n$.

In particular cases, the Bernstein operators can be obtained by setting $s = 0$ or $s = 1$ with $r = 0$ or $s = 1$ with $r = 1$.

1.5 Kantorovich Operators

Kantorovich operators are an important class of approximation operators used in the theory of function approximation. These operators are specifically designed for functions that are integrable and have the remarkable property of preserving the integral $\int_0^1 f(t)\,dt$ when applied to a function f to be approximated.

The Kantorovich operators are defined as follows:

$$K_m : L_1([0,1]) \to C([0,1]),$$
$$K_m(f)(x) = (m+1) \sum_{k=0}^{m} b_{m,k}(x) \int_{k/(m+1)}^{(k+1)/(m+1)} f(t)\,dt,$$

where $b_{m,k}(x)$ are Bernstein polynomials given by:

$$b_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}.$$

Theorem 1.5.1. *(Agratini, [4])*
Kantorovich operators have the following properties:

i)
$$K_m(e_0)(x) = 1,$$
where $e_0(x) = 1$.

ii)
$$K_m(e_1)(x) = \frac{m}{m+1}x + \frac{1}{2(m+1)},$$
where $e_1(x) = x$.

iii)
$$K_m(e_2)(x) = \frac{m(m-1)}{(m+1)^2}x^2 + \frac{2m}{(m+1)^2}x + \frac{1}{3(m+1)^2},$$
where $e_2(x) = x^2$.

iv)
$$\lim_{m\to\infty} K_m(f)(x) = f(x) \text{ uniformly on } [0,1] \text{ for all } f \in C([0,1]),$$
and
$$\lim_{m\to\infty} K_m(f)(x) = f(x) \text{ for all } f \in L_p([0,1]), p \geq 1.$$

v) If $f \in C([0,1])$ *and* $x \in [0,1]$*, then*
$$|K_m(f)(x) - f(x)| \leq 2\omega\left(f, \frac{1}{2\sqrt{m+1}}\right),$$
where $\omega(f,\delta)$ *denotes the modulus of continuity of* f*.*

1.6 Durrmeyer operators

Durrmeyer operators, discovered by Durrmeyer and independently by Lupaș ([39]), are a modification of Bernstein operators with notable properties related to convergence, approximation, and spectral structure. They are particularly well-regarded for their ability to preserve certain integral properties and to address differential equations.

For $m \in \mathbb{N}$, the Durrmeyer operators are defined by:

$$D_m : L_1([0,1]) \to C([0,1]),$$
$$D_m(f)(x) = (m+1)\sum_{k=0}^{m} b_{m,k}(x)\int_0^1 b_{m,k}(t)f(t)\,dt,$$

where $b_{m,k}(x)$ are Bernstein polynomials defined as:

$$b_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}.$$

The Durrmeyer operators offer several significant properties:

Theorem 1.6.1. *(Derriennic, [24])*
The Durrmeyer operators have the following properties:

i)
$$D_m(e_0)(x) = 1;$$

ii) Durrmeyer operators transform any polynomial of degree p, where $p \leq m$, into a polynomial of the same degree.

iii)
$$\lim_{m\to\infty} D_m(f)(x) = f(x) \text{ uniformly on } [0,1] \text{ for all } f \in C([0,1]).$$

iv) If f and g are integrable functions on $[0,1]$, then

$$\int_0^1 D_m(f)(x)g(x)\,dx = \int_0^1 f(t)D_m(g)(t)\,dt \text{ for all } m \in \mathbb{N}.$$

v) The Legendre polynomials are eigenfunctions of the operator D_m for all $m \in \mathbb{N}$. The eigenvalue associated with the Legendre polynomial of degree n is given by:

$$\lambda_{m,n} = \begin{cases} \dfrac{(m+1)!m!}{(m-n)!(m+n+1)!}, & \text{if } n \leq m, \\ 0, & \text{if } n > m. \end{cases}$$

vi) If f is integrable and bounded on $[0,1]$ and twice differentiable at $x \in [0,1]$, then

$$\lim_{m\to\infty} m(D_m(f)(x) - f(x)) = (1-2x)f'(x) + x(1-x)f''(x).$$

vii) If f is integrable and bounded on $[0,1]$ and admits a derivative of order r at $x \in [0,1]$, then

$$\lim_{m\to\infty} \frac{d^r}{dx^r} D_m(f)(x) = \frac{d^r}{dx^r} f(x).$$

viii) If $f \in C([0,1])$ and $m \geq 3$, then

$$|D_m(f)(x) - f(x)| \leq 2\omega\left(f, \frac{1}{\sqrt{2m+6}}\right),$$

.

1.7 Beta operators

The Beta function $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$, where $x,y>0$ and $t\in[0,1]$, is used as a weight function in defining Beta operators.

Beta operators, introduced by A. Lupaș ([39]), are defined as follows:

$$\mathbb{B}_n(f)(x)=\int_0^1 \frac{t^{nx}(1-t)^{n(1-x)}}{B(nx+1,n+1-nx)}f(t)\,dt,\ \ f\in C([0,1]),\ x\in[0,1], \tag{1.9}$$

where $B(u,v)$ denotes the Beta function.

Theorem 1.7.1. *(Lupaș, [39]) Beta operators have the following properties:*

i) $(\mathbb{B}_n e_p)(x)=\frac{(nx+p)(nx+p-1)\cdots(nx+1)}{(n+2)(n+3)\cdots(n+p+1)}$;

ii) $\lim_{n\to\infty}\mathbb{B}_n(f)(x)=f(x)$ *uniformly for all* $f\in C([0,1])$.

1.8 Balázs operators

The Balázs operators can be used to approximate a function over the unbounded interval $[0,\infty)$. For $f\in C([0,\infty))$, the Balázs operators ([8]) are defined as:

$$\begin{aligned}(R_nf)(x) &= \frac{1}{(1+a_nx)^n}\sum_{j=0}^{n}\binom{n}{j}(a_nx)^j f\left(\frac{j}{b_n}\right) \\ &= \sum_{j=0}^{n}p_{n,j}\left(\frac{a_nx}{1+a_nx}\right)f\left(\frac{j}{b_n}\right),\quad x\geq 0,\ n\in\mathbb{N},\end{aligned} \tag{1.10}$$

where

$$p_{n,j}(z)=\binom{n}{j}z^j(1-z)^{n-j},\quad z\geq 0,$$

and $\{a_n\}_n$, $\{b_n\}_n$ are two sequences of positive real numbers chosen appropriately.

Theorem 1.8.1. *(Balázs, [8])*

The Balázs operators have the following properties:

i) $\frac{1}{1+\alpha_nx}\sum_{j=0}^{n}\binom{n}{j}(\alpha_nx)^j=1$;

ii) $\frac{1}{1+\alpha_nx}\sum_{j=0}^{n}(j-b_nx)\binom{n}{j}(\alpha_nx)^j=\frac{-\alpha_nb_nx^2}{1+\alpha_nx}$;

iii) $\frac{1}{1+\alpha_nx}\sum_{j=0}^{n}(j-b_nx)^2\binom{n}{j}(\alpha_nx)^j=\frac{\alpha_n^2b_n^2x^4+b_nx}{(1+\alpha_nx)^2}$, *for all* $x\geq 0$ *and* $\alpha_n=\frac{b_n}{n}$, $b_n>0$.

Chapter 2

Construction of new types of approximation operators

In this chapter, we will present several methods for constructing new approximation operators. The chapter is organized into three sections, each corresponding to a method for constructing these operators. The first section is dedicated to construction using Boolean sums, the second section focuses on the construction of a new operator based on a modification of Bernstein operators, and the last section highlights a Stancu-type generalization.

2.1 Operators constructed using boolean sums

This section contains results presented by the author in [40].

By using combinations of linear and positive operators, it is possible to improve the approximation order of Beta operators. One method is the use of Boolean sums. The Boolean sum of two operators S and T is defined as the operator $S+T-S\circ T$.

Consider the sequence of operators $(L_n)_n$, defined as follows:

$$L_n := 2\mathbb{B}_n - \mathbb{B}_n \circ \mathbb{B}_n. \tag{2.1}$$

To study the operator L_n, we need to calculate the moments of the operators $\mathbb{B}_n$.

For $k = 0, 1, 2, \ldots$, $n \geq 1$, and $x \in [0,1]$, we denote:

$$m_{n,k}(x) = \mathbb{B}_n((e_1 - xe_0)^k)(x).$$

Using the previous formulas and the function $\Psi(x) = x(1-x)$, we obtain:

Lemma 2.1.1. *[40] We have:*

$$m_{n,1}(x) = A_{n,1}\Psi'(x) \tag{2.2}$$
$$m_{n,2}(x) = A_{n,2}\Psi(x) + B_{n,2} \tag{2.3}$$
$$m_{n,3}(x) = A_{n,3}\Psi(x)\Psi'(x) + B_{n,3}\Psi'(x) \tag{2.4}$$
$$m_{n,4}(x) = A_{n,4}\Psi(x)^2 + B_{n,4}\Psi(x) + C_{n,4}, \tag{2.5}$$

where

$$A_{n,1} = \frac{1}{n+2},\ A_{n,2} = \frac{n-6}{(n+2)(n+3)},\ B_{n,2} = \frac{2}{(n+2)(n+3)} \tag{2.6}$$
$$A_{n,3} = \frac{5n-12}{(n+2)(n+3)(n+4)},\ B_{n,3} = \frac{6}{(n+2)(n+3)(n+4)} \tag{2.7}$$
$$A_{n,4} = \frac{3n^2-86n+120}{(n+2)(n+3)(n+4)(n+5)} \tag{2.8}$$
$$B_{n,4} = \frac{26n-120}{(n+2)(n+3)(n+4)(n+5)} \tag{2.9}$$
$$C_{n,4} = \frac{24}{(n+2)(n+3)(n+4)(n+5)}. \tag{2.10}$$

Additionally,

$$m_{n,6}(x) = O\left(\frac{1}{n^3}\right). \tag{2.11}$$

Theorem 2.1.1. *[40] The operators L_n satisfy the following condition: If $f \in C([0,1])$, then:*

$$\lim_{n\to\infty} L_n(f)(x) = f(x),\ \textit{uniformly for } x \in [0,1].$$

We use Boolean sums of Beta operators to achieve an improvement in the approximation order expressed by the Voronovskaja-type theorem. Thus, we obtain a sequence of linear and positive operators $(L_n)_n$, where $L_n : C([0,1]) \to C([0,1])$, such that there exists α_n with $\frac{\alpha_n}{n} \to \infty$ so that the relation (1.1) is satisfied.

For Beta operators, the following result is known.

Theorem 2.1.2. *(Lupaș, [39]) If $f \in C^{IV}([0,1])$, for each $x \in [0,1]$, we have:*

$$\lim_{n\to\infty} n[\mathbb{B}_n(f)(x) - f(x)] = (1-2x)f'(x) + \frac{1}{2}x(1-x)f''(x). \tag{2.12}$$

Therefore, Beta operators satisfy the Voronovskaja theorem for $\alpha_n = n$. To achieve a better approximation order in the Voronovskaja theorem, consider the sequence of operators $(L_n)_n$, defined in (2.1).

Theorem 2.1.3. *[40] If $f \in C^4([0,1])$ and $x \in [0,1]$, then:*

$$\begin{aligned}\lim_{n\to\infty} n^2[L_n(f)(x) - f(x)] &= 2f'(x)\Psi'(x) + f''(x)\left[-\frac{3}{2} + 8\Psi(x)\right] \\ &\quad -2f'''(x)\Psi(x)\Psi'(x) - \frac{1}{2}f^{IV}(x)\Psi^2(x).\end{aligned}$$

2.2 Generalized Bernstein operators obtained by truncating the domain

This section presents results that the author has discussed in the paper [41].

In [21], the following Bernstein-type operators were introduced. We denote by V_n the operators defined as:

$$V_n(f)(x) = \sum_{k=0}^{n} b_{n,k}(x) \cdot f\left(\frac{k}{n}\right)$$

where

$$b_{n,k}(x) = \left(1 + \frac{1}{n}\right)^n \cdot \binom{n}{k} \cdot x^k \cdot \left(\frac{n}{n+1} - x\right)^{n-k}$$

and

$$f \in C([0,1]),\ x \in \left[0, 1 - \frac{1}{n+1}\right],\ n \in \mathbb{N}.$$

Similarly, we introduce the following generalized operators [41]:

$$S_n^t : C([0,1]) \to C\left[0, \frac{n}{n+t}\right],\ t > 0,\ n \in \mathbb{N},$$

$$S_n^t(f)(x) = \sum_{k=0}^{n} s_{n,k}(x) \cdot f\left(\frac{k}{n+t}\right),\ \forall f \in C([0,1]),\ \ x \in \left[0, \frac{n}{n+t}\right], \tag{2.13}$$

where

$$s_{n,k}(x) = \left(\frac{n+t}{n}\right)^n \cdot \binom{n}{k} \cdot x^k \cdot \left(\frac{n}{n+t} - x\right)^{n-k}.$$

The operators S_n^t can also be defined in the space $C\left[0, \frac{n}{n+t}\right]$, and we use the same symbol for the function $f \in C([0,1])$ and its restriction to the interval $\left[0, \frac{n}{n+t}\right]$.

We use the Pochhammer symbol: $(a)_r = a(a-1)\dots(a-r+1)$, for $a \in \mathbb{R}$, $r \in \mathbb{N}$.

Proposition 2.2.1. *[41] The operators S_n^t satisfy the following properties:*

i) $S_n^t(f)(x) \geq 0$, *if* $f \in C\left[0, \frac{n}{n+t}\right]$, $f \geq 0$;

ii) $S_n^t(e_0)(x) = 1$;

iii) $S_n^t(e_1)(x) = x$;

iv) $S_n^t(e_2)(x) = x\left(\frac{n-1}{n} \cdot x + \frac{1}{n+t}\right)$,

where $x \in \left[0, 1 - \frac{1}{n+t}\right]$.

Proposition 2.2.2. *[41] The following recurrence relation holds:*

$$x\left(\frac{n}{n+t} - x\right) s'_{n,k}(x) = n \cdot \left(\frac{k}{n+t} - x\right) s_{n,k}(x).$$

Theorem 2.2.1. *[41] Let $\mu_{n,m}(x)$ be the m-th moment for the operators defined in (2.13) as follows:*

$$\mu_{n,m}(x) = \sum_{k=0}^{n} s_{n,k}(x)\left(\frac{k}{n+t} - x\right)^m, \; m = 0, 1, 2, ...$$

Then we have:

i) $\mu_{n,0}(x) = 1$;

ii) $\mu_{n,1}(x) = 0$;

iii) $n\mu_{n,m+1}(x) = x\left(\frac{n}{n+t} - x\right)\left(\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right)$;

iv) $\mu_{n,2}(x) = \frac{x}{n}\left(\frac{n}{n+t} - x\right)$;

v) $\mu_{n,3}(x) = \frac{x}{n}\left(\frac{n}{n+t} - x\right)\left(\frac{1}{n+t} - \frac{2x}{n}\right)$;

vi) $\mu_{n,4}(x) = \frac{x}{n}\left(\frac{n}{n+t} - x\right)\left[\frac{1}{n}\left(\frac{n}{n+t} - x\right)\left(\frac{1}{n+t} - \frac{4x}{n} + 3\right) - \frac{x}{n}\left(\frac{1}{n+t} - \frac{2x}{n}\right)\right]$.

Theorem 2.2.2. *[41] For any $t > 0$, $f \in C([0,1])$, and $0 < \epsilon < 1$, the following relation holds:*

$$\lim_{n\to\infty} S_n^t(f)(x) = f(x)$$

uniformly on the interval $[0, 1-\epsilon]$.

Theorem 2.2.3. *[41] Let $f : [0,1] \to \mathbb{R}$ be a bounded function that is twice differentiable at $x \in (0,1)$. Then:*

$$\lim_{n\to\infty} n\left[S_n^t(f)(x) - f(x)\right] = \frac{x(1-x)}{2} f''(x).$$

We recall the following well-known definitions:

Definition 2.2.1. *A function $g : I \to \mathbb{R}$, where I is an interval, is called convex of order $r \geq -1$ if all divided differences over $r+2$ points on the interval I are non-negative.*

Thus, a positive function is a convex function of order -1, an increasing function is convex of order 0, and so forth. In other words, for $r \geq 0$, if $f \in C^{r+1}(I)$, then f is convex of order r if and only if $f^{(r+1)} \geq 0$.

Definition 2.2.2. *A linear operator is called a convex operator of order r, $r \geq -1$, if it maps any r-convex function to an r-convex function.*

The following property is essential for proving the existence of simultaneous approximation.

Theorem 2.2.4. *[41] The operators S_n^t are convex of order $r-1$, $\forall r \in [0,n]$.*

The study of simultaneous approximation is based on using higher-order Kantorovich operators.

Theorem 2.2.5. *[41] Let $T_{n,r}(x) = S_n^t(e_r)(x)$. Then:*

$$T_{n,r+1}(x) = x \cdot T_{n,r}(x) + \frac{x}{n}\left(\frac{n}{n+t} - x\right) T'_{n,r}(x).$$

Theorem 2.2.6. *[41] For $n \geq 1$, $r \geq 0$, and $x \in [0,1]$, we have:*

$$T_{n,r}(x) = A_{n,r}x^r + B_{n,r}x^{r-1} + C_{n,r}x^{r-2} + R_{n,r}(x)$$

where

$$\begin{aligned} A_{n,r} &= \frac{(n-1)_{r-1}}{n^{r-1}}, \\ B_{n,r} &= \frac{r(r-1)}{2} \cdot \frac{(n-1)_{r-2}}{n^{r-2}(n+t)}, \\ C_{n,r} &= \frac{r(r-1)(r-2)(3r-5)}{24} \cdot \frac{(n-1)_{r-3}}{n^{r-3}(n+t)^2} \end{aligned}$$

and $R_{n,r}$ is a polynomial of degree $r-3$.

The main result is as follows:

Theorem 2.2.7. *[41] For any function $f \in C^r([0,1])$, $r \geq 1$, any $t > 0$, and $\varepsilon > 0$, we have:*

$$\lim_{n\to\infty} (S_n^t(f)(x))^{(r)} = f^{(r)}(x), \text{ uniformly for } x \in [0, 1-\varepsilon]. \tag{2.14}$$

2.3 Balázs-Stancu Type Operators

This section contains results presented by the author in the paper [45], in collaboration with Maria Talpău Dimitriu.

The Balázs operators introduced in Section 1.9 have been studied and generalized in several directions: [9], [68], [10], [1], [5], [6], [31].

In this section, we consider a generalization of the Balázs operators in the manner of the generalization of Bernstein operators introduced by D. D. Stancu in [64]:

$$(S_{n,r,s}f)(x) = \sum_{j=0}^{n-rs} p_{n-rs,j}(x) \sum_{i=0}^{s} p_{s,i}(x) f\left(\frac{j+ir}{n}\right), \tag{2.15}$$

where $f \in C([0,1])$, $x \in [0,1]$, and $n \in \mathbb{N}$, with $r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ fixed such that $rs < n$.

Consider the Balázs-Stancu operators defined as follows:

$$(R_{n,r,s}f)(x) = \sum_{j=0}^{n-rs} p_{n-rs,j}\left(\frac{a_n x}{1+a_n x}\right) \sum_{i=0}^{s} p_{s,i}\left(\frac{a_n x}{1+a_n x}\right) f\left(\frac{j+ir}{na_n}\right), \tag{2.16}$$

where $f \in C[0,\infty)$, $x \geq 0$, $n \in \mathbb{N}$, $r, s \in \mathbb{N}_0$ such that $rs < n$, and $(a_n)_n$ is a sequence of positive real numbers.

If $a_n = 1$ for all $n \in \mathbb{N}$, then $(R_{n,r,s}f)(x) = (S_{n,r,s}f)\left(\frac{x}{1+x}\right)$.

Lemma 2.3.1. *[45] The operator $S_{n,r,s}$ satisfies the following relations:*

(i) $(S_{n,r,s}e_0)(x) = 1$;

(ii) $(S_{n,r,s}e_1)(x) = x$;

(iii) $(S_{n,r,s}e_2)(x) = x^2 + \left(1 + \frac{rs(r-1)}{n}\right) \cdot \frac{x(1-x)}{n}$,

where $x \in [0,\infty)$ and $e_i(y) = y^i$, for $i = 0,1,2$.

Lemma 2.3.2. *[45] The operator $R_{n,r,s}$ satisfies the following relations:*

(i) $R_{n,r,s}f \geq 0$, *for all* $f \in C[0,\infty)$, *with* $f \geq 0$;

(ii) $(R_{n,r,s}e_0)(x) = 1$;

(iii) $(R_{n,r,s}e_1)(x) = \frac{x}{1+a_n x}$;

(iv) $(R_{n,r,s}e_2)(x) = \frac{x^2}{(1+a_n x)^2} + \left(1 + \frac{rs(r-1)}{n}\right) \cdot \frac{x}{na_n(1+a_n x)^2}$,

where $x \in [0,\infty)$ and $e_p(y) = y^p$, for $p = 0,1,2$.

Lemma 2.3.3. *[45] Let the m-th moment for the operator be denoted as follows:*

$$\left(R_{n,r,s}(e_1 - xe_0)^m\right)(x) = \sum_{j=0}^{n-rs} p_{n-rs,j}\left(\frac{a_n x}{1+a_n x}\right) \sum_{i=0}^{s} p_{s,i}\left(\frac{a_n x}{1+a_n x}\right) \left(\frac{j+ir}{na_n} - x\right)^m, \ m = 1,2,\ldots.$$

Then we have:

(i)

$$\left(R_{n,r,s}(e_1 - xe_0)\right)(x) \;=\; -\frac{a_n x^2}{1+a_n x};$$

(ii)

$$\left(R_{n,r,s}(e_1 - xe_0)^2\right)(x) \;=\; r\frac{a_n^2 x^4}{(1+a_n x)^2} + \left(1 + \frac{rs(r-1)}{n}\right)\cdot \frac{x}{na_n(1+a_n x)^2}.$$

Theorem 2.3.1. *[45] If* $\liminf_n a_n = 0$ *and* $\liminf_n na_n = \infty$*, then for a bounded function* $f \in UC_B([0,\infty))$*, it follows that*

$$\lim_{n\to\infty} R_{n,r,s} f = f \text{ uniformly on any compact interval } K \subset [0,\infty).$$

Recall that the modulus of continuity for the continuous function f on $[0,\infty)$ is defined by:

$$\omega(f,t) = \sup\left\{|f(y) - f(x)| : x, y \in [0,\infty), |y-x| \le t\right\},\ t > 0,$$

provided this modulus is bounded.

Theorem 2.3.2. *[45] For any function* $f \in C[0,\infty)$ *such that* $\omega(f,t) < \infty$ *for all* $t > 0$*, the following inequality holds:*

$$|(R_{n,r,s} f)(x) - f(x)| \le 2\omega\left(f, \theta_{n,r,s,x}\right), \tag{2.17}$$

where

$$\theta_{n,r,s,x} = \sqrt{\frac{a_n^2 x^4}{(1+a_n x)^2} + \left(1 + \frac{rs(r-1)}{n}\right)\cdot \frac{x}{na_n(1+a_n x)^2}}.$$

Remark 2.3.1. For $f \in C[0,\infty)$ and $M > 0$, we have

$$\|R_{n,r,s} f - f\|_{[0,M]} \le 2\omega\left(f, \sqrt{a_n^2 M^4 + \left(1 + \frac{rs(r-1)}{n}\right)\cdot \frac{M}{na_n}}\right). \tag{2.18}$$

Corollary 2.3.1. *[45] If* f *is a function that is uniformly continuous and bounded on* $[0,\infty)$*, then* f *can be uniformly approximated on any compact interval* $K \subset [0,\infty)$*.*

We will now present the preservation of monotone and convex functions by the constructed operators.

Lemma 2.3.4. *[45] For $f \in C([0,\infty))$, $0 \le x < y$, and $\lambda \in [0,1]$, we have*

$$
\begin{aligned}
&(R_{n,r,s}f)\left((1-\lambda)x+\lambda y\right)\\
&= \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}\left(\frac{a_n x}{1+a_n x}, \frac{a_n(y-x)}{(1+a_n x)(1+a_n y)}\right) \times\\
&\times p_{n-rs,k_2,l_2}\left(\frac{a_n x}{1+a_n x}, \frac{a_n(y-x)}{(1+a_n x)(1+a_n y)}\right) \times\\
&\times \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}\left(\frac{\lambda(1+a_n y)}{\lambda(1+a_n y)+(1-\lambda)(1+a_n x)}\right) \times\\
&\times p_{l_2,m_2}\left(\frac{\lambda(1+a_n y)}{\lambda(1+a_n y)+(1-\lambda)(1+a_n x)}\right) f\left(\frac{k_2+m_2+r(k_1+m_1)}{n a_n}\right),
\end{aligned}
$$

where $p_{m,k,l}(u,v) = \frac{m!}{k!l!(m-k-l)!}u^k v^l (1-u-v)^{m-k-l}$ is the Bernstein basis for the space of polynomials in two variables.

Theorem 2.3.3. *[45] Let $f \in C([0,\infty))$. If f is a non-increasing function, then $R_{n,r,s}f$ is also a non-increasing function.*

Theorem 2.3.4. *[45] Let $f \in C([0,\infty))$ be a non-increasing function. If f is a convex function, then $R_{n,r,s}f$ is also a convex function.*

At the end of this section, we will present the invariance property of certain classes of Lipschitz functions, following the model in [66].

Let Lip_M^α denote the class of Lipschitz functions on $[0,\infty)$ with exponent $\alpha \in [0,1]$ and Lipschitz constant $M > 0$, i.e., the set of all continuous real-valued functions f defined on $[0,\infty)$ that satisfy the condition:

$$|f(x) - f(y)| \le M \cdot |x-y|^\alpha, \quad \forall x, y \in [0,\infty).$$

Theorem 2.3.5. *[45] Let $f \in C([0,\infty))$, $M > 0$, and $\alpha \in [0,1]$. If $f \in Lip_M^\alpha$, then $R_{n,r,s}f \in Lip_M^\alpha$.*

Chapter 3

Properties of $\mathbb{B}$-concavity and $\mathbb{BB}$-concavity in connection with Bernstein operators

This chapter contains results presented by the author in [43] and a paper submitted for publication [46].

The starting point is the work of [15] and [69], where the concepts of $\mathbb{B}$-convexity and $\mathbb{B}$-concavity for functions of several variables were introduced. They also demonstrated that increasing and $\mathbb{B}$-concave functions are transformed by Bernstein operators into functions of the same type.

Among other classes of functions preserved by Bernstein operators are: higher-order convex (concave) functions (Popoviciu, [55]), Lipschitz classes of order one (Lindvall, [36]), Lipschitz classes associated with a modified second-order modulus (Zhou, [73]), and higher-order quasiconvexity (Păltănea, [53]).

In this chapter, we introduce the concept of $\mathbb{BB}$-concavity, which represents a slight modification of $\mathbb{B}$-concavity and provide characterizations of functions that possess this property. The central objective is to demonstrate that $\mathbb{BB}$-concave and increasing functions defined on an s-dimensional simplex are transformed into $\mathbb{BB}$-concave and increasing functions by s-dimensional Bernstein operators on a simplex.

In this way, we generalize the notions introduced and results established in [43], where we introduced the concept of $\mathbb{BB}$-concavity for functions of two variables and showed a connection between these types of concavity and two-dimensional Bernstein operators.

3.1 $\mathbb{B}$-concavity and $\mathbb{BB}$-concavity

For $\overline{a} = (a_1, a_2, ..., a_s)$ and $\overline{b} = (b_1, b_2, ..., b_s)$ in $\mathbb{R}^s$, define

$$\overline{a} \vee \overline{b} := (\max\{a_1, b_1\}, \max\{a_2, b_2\}, ..., \max\{a_s, b_s\}).$$

Definition 3.1.1. *(Briec, Horvath, [15]), A function* $f : M \subset \mathbb{R}^s_+ \to (0, \infty)$ *is called* $\mathbb{B}$*-concave if it satisfies the following properties:*

i) $\lambda\overline{a} \vee \overline{b} \in M, \forall \overline{a}, \overline{b} \in M, \lambda \in [0,1]$;

ii) $f(\lambda\overline{a} \vee \overline{b}) \geq \lambda f(\overline{a}) \vee f(\overline{b}), \forall \overline{a}, \overline{b} \in M, \lambda \in [0,1]$.

We define a partial order $'' \leq ''$ on $\mathbb{R}^s_+$ by: $a \leq b$ if and only if $a_i \leq b_i, \forall i = \overline{1, s}$, where $\overline{a} = (a_1, a_2, ..., a_s)$ and $\overline{b} = (b_1, b_2, ..., b_s)$.

A function $f : D \subset \mathbb{R}^s \to \mathbb{R}$ is called increasing if $f(\overline{a}) \leq f(\overline{b}), \forall \overline{a}, \overline{b} \in D, \overline{a} \leq \overline{b}$.

Theorem 3.1.1. *[46] Let* $M \subset \mathbb{R}^s$ *be a domain that satisfies condition i) from Definition 3.1.1 and* $\overline{0} \in M$*. If* $f : M \longrightarrow (0, \infty)$ *is an increasing function, then the following statements are equivalent:*

i) f *is* $\mathbb{B}$*-concave.*

ii) For all vectors $\overline{v} = (v_1, v_2, \ldots, v_s) \in M$*, the function*

$$\varphi_v : [0,1] \longrightarrow \mathbb{R}, \quad \varphi_v(t) = \frac{t}{f(t\overline{v})}, \quad t \in [0,1]$$

is increasing.

Theorem 3.1.2. *[46] Let* $f : M \to \mathbb{R}^*_+$ *be an increasing and differentiable function in all s variables, where* $M \subset \mathbb{R}^s$ *is a domain that satisfies condition i) from Theorem 3.1.1. The inequality*

$$f(t\overline{v}) - t \cdot \sum_{i=1}^{s} \frac{\partial f(tv)}{\partial x_i} \cdot v_i \geq 0 \tag{3.1}$$

holds for all $\overline{v} \in M$*, for all* $t \in [0,1]$*, and* $\overline{0} \in M$*, where* $\overline{v} = (v_1, v_2, \ldots, v_s)$*, if and only if* f *is* $\mathbb{B}$*-concave. The inequality (3.1) is equivalent to:*

$$f(\overline{x}) - \sum_{i=1}^{s} x_i \cdot \frac{\partial f}{\partial x_i}(\overline{x}) \geq 0, \quad \forall \overline{x} \in M. \tag{3.2}$$

Indeed, we can take $\overline{v} = (x_1, x_2, \ldots, x_s)$ and $t = 1$.

It is straightforward to show that Δ satisfies condition i) from Definition 3.1.1 and $\overline{0} \in \Delta$.

Definition 3.1.2. *If* $(x_1, \ldots, x_s) \in \Delta$ *and* $(y_1, \ldots, y_s) \in \Delta$*, we define* $\overline{y} \prec \overline{x}$ *if for all i,* $y_i < x_i$ *when* $x_i > 0$ *and* $y_i = x_i$ *when* $x_i = 0$*.*

Definition 3.1.3. *A function* $f : \Delta \to \mathbb{R}$ *is* $\mathbb{BB}$*-concave if for all* $\overline{x} \in \Delta$ *with at least two nonzero components and for all* $\overline{y} \in \Delta$ *such that* $\overline{y} \prec \overline{x}$*, we have:*

$$f(\overline{x}) \leq \sum_{i=1}^{s} \alpha_i f(\overline{z_i}), \tag{3.3}$$

where $\overline{z_i} = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_s)$, $i = \overline{1, s}$, *and* α_i *are the solutions of the system of equations:*

$$\overline{x} = \sum_{i=1}^{s} \alpha_i \overline{z_i}. \tag{3.4}$$

Proposition 3.1.1. *[46] If* $\overline{x} \in \Delta$ *with at least two nonzero components and* $\overline{y} \in \Delta$ *with* $\overline{y} \prec \overline{x}$, *the solutions for (3.4) are:*

$$\begin{cases} \alpha_i = \dfrac{x_i}{x_i - y_i} \left(\displaystyle\sum_{j \in Q} \dfrac{x_j}{x_j - y_j} - 1 \right)^{-1}, & i \in Q \\ \alpha_i = 0, & i \in \{1, \ldots, s\} - Q \end{cases} \tag{3.5}$$

where $Q = \{i \in \{1, \ldots, s\} \mid x_i > y_i\} = \{i \in \{1, \ldots, s\} \mid x_i > 0\}$.

Remark 3.1.1. In the two-dimensional case, consider:

$$\Delta = \{(x, y) \mid x \geq 0, y \geq 0, x + y \leq 1\}. \tag{3.6}$$

A function $f : \Delta \to \mathbb{R}$ is $\mathbb{BB}$-concave if the following inequality holds:

$$f(x, y) - \alpha f(z, y) - \beta f(x, w) \leq 0, \tag{3.7}$$

for all $(x, y) \in \Delta$, $(z, y) \in \Delta$, $(x, w) \in \Delta$, such that $z \leq x$, $w \leq y$, and $z \cdot w < x \cdot y$, where

$$\alpha = \frac{x \cdot y - x \cdot w}{x \cdot y - z \cdot w} \text{ and } \beta = \frac{x \cdot y - y \cdot z}{x \cdot y - z \cdot w}. \tag{3.8}$$

From the condition $z \cdot w < x \cdot y$, it follows that $x, y > 0$, and thus Proposition 3.1.1 applies.

Thus, the numbers α and β are unique coefficients for which:

$$(x, y) = \alpha \cdot (z, y) + \beta \cdot (x, w).$$

Therefore, a function $f : \Delta \to \mathbb{R}$ is $\mathbb{BB}$-concave if the following inequality holds:

$$f\Big(\alpha \cdot (z, y) + \beta \cdot (x, w)\Big) \leq \alpha f(z, y) + \beta f(x, w), \tag{3.9}$$

for all $(x, y) \in \Delta$, $(z, y) \in \Delta$, $(x, w) \in \Delta$, and $z \leq x$, $w \leq y$, $z \cdot w < x \cdot y$, where $\alpha \in [0, 1]$ and $\beta \in [0, 1]$ are numbers given by (3.8) and Δ is the domain indicated in (3.6).

Theorem 3.1.3. *[46] If* $f : \Delta \to \mathbb{R}$ *is a differentiable,* $\mathbb{BB}$*-concave function and* $f(\overline{0}) \geq 0$, *then the following inequality holds:*

$$f(\overline{x}) - \sum_{i=1}^{s} x_i \cdot \frac{\partial f}{\partial x_i}(\overline{x}) \geq 0, \quad \forall \overline{x} \in \Delta. \tag{3.10}$$

Corollary 3.1.1. *[46] If $f : \Delta \to (0,\infty)$ is an increasing, differentiable, and $\mathbb{BB}$-concave function, then f is $\mathbb{B}$-concave.*

3.2 $\mathbb{BB}$-concavity of Bernstein operators for the s-dimensional simplex

Let $n \in \mathbb{N}$ and consider the Bernstein operators B_n of s variables defined on $C(\Delta)$ as:

$$B_n(f)(\overline{x}) = \sum_{\overline{k}\in\Lambda_n} p_{n,\overline{k}}(\overline{x}) \cdot f\Big(\frac{\overline{k}}{n}\Big), \quad f : \Delta \to \mathbb{R}, \quad \overline{x} \in \Delta \tag{3.11}$$

where

$$p_{n,\overline{k}}(\overline{x}) = \binom{n}{\overline{k}} \cdot \overline{x}^{\overline{k}} \cdot (1 - |\overline{x}|)^{n-|\overline{k}|}, \quad \overline{k} \in \Lambda_n, \quad \overline{x} \in \Delta. \tag{3.12}$$

In the case of $s = 1$,

$$p_{r,q}(z) = \binom{r}{q} \cdot z^q \cdot (1 - z)^{r-q}.$$

By convention, $p_{r,q}(z) = 0$ if $q < 0$ or $q > r$.

Proposition 3.2.1. *[46]*
The equation (3.11) can be rewritten for $\overline{x} \in \Delta$ such that $x_1 + \ldots + x_{s-1} < 1$ in the form:

$$B_n(f)(\overline{x}) = \sum_{k_1=0}^{n} p_{n,k_1}(x_1) \sum_{k_2=0}^{n-k_1} p_{n-k_1,k_2}\Big(\frac{x_2}{1-x_1}\Big) \cdots$$

$$\times \sum_{k_s=0}^{n-k_1-\ldots-k_{s-1}} p_{n-k_1-\ldots-k_{s-1},k_s}\Big(\frac{x_s}{1-x_1-\ldots-x_{s-1}}\Big) f\Big(\frac{\overline{k}}{n}\Big). \tag{3.13}$$

Theorem 3.2.1. *[46] If $f : \Delta \to \mathbb{R}$ is an increasing and $\mathbb{BB}$-concave function and if $f(\overline{0}) \geq 0$, then*

$$B_n(f)(\overline{x}) - \sum_{i=1}^{s} x_i \cdot \frac{\partial}{\partial x_i} B_n(f)(\overline{x}) \geq 0, \quad \overline{x} \in \Delta.$$

Corollary 3.2.1. *[46]*
If $f : \Delta \to (0,\infty)$ is an increasing $\mathbb{BB}$-concave function, then $B_n(f)$ is $\mathbb{B}$-concave.

Remark 3.2.1. In the case of $s = 2$, consider the Bernstein operators B_n of two variables defined on $C(\Delta)$ by:

$$B_n(f)(x,y) = \sum_{k\geq 0, l\geq 0, k+l\leq n} b_{n,k,l}(x,y) \cdot f\Big(\frac{k}{n}, \frac{l}{n}\Big), \quad f \in C(\Delta), \quad (x,y) \in \Delta \tag{3.14}$$

where:

$$b_{n,k,l}(x,y) = \frac{n!}{k!l!(n-k-l)!} \cdot x^k \cdot y^l \cdot (1-x-y)^{n-k-l}. \tag{3.15}$$

The equation (3.14) can be rewritten as:

$$B_n(f)(x,y) = \sum_{k=0}^{n} b_{n,k}(x) \sum_{l=0}^{n-k} b_{n-k,l}\Big(\frac{y}{1-x}\Big) \cdot f\Big(\frac{k}{n}, \frac{l}{n}\Big), \tag{3.16}$$

where:

$$b_{r,s}(z) = \frac{r!}{s!(r-s)!} \cdot z^s \cdot (1-z)^{r-s}, \quad z \in [0,1]. \tag{3.17}$$

If $f : \Delta \to \mathbb{R}$ is $\mathbb{BB}$-concave and increasing, then:

$$B_n(f)(x,y) - x \cdot \frac{\partial}{\partial x} B_n(f)(x,y) - y \cdot \frac{\partial}{\partial y} B_n(f)(x,y) \geq 0.$$

If $f \in C(\Delta)$ is $\mathbb{BB}$-concave and increasing in both variables, then $B_n(f)$ is $\mathbb{B}$-concave.

Chapter 4

Properties of a class of modified Stancu Operators constructed with the Pólya distribution

This chapter contains results presented by the author in the paper [44], in collaboration with Mihai N. Pascu and Nicolae R. Pascu.

In his work, Bernstein [13] provided a simple proof of the Weierstrass theorem on the uniform approximation of continuous functions by polynomials, known as Bernstein polynomials.

Later, D. D. Stancu observed that the binomial distribution used by Bernstein is a special case of the Pólya urn distribution, where the replacement parameter is equal to zero. Stancu introduced in [61] and [62] a more general class of operators, known in the literature as Pólya-Stancu operators, denoted by P_n^c (see (4.7)).

A first observation is that if we choose the replacement parameter c to be a negative real number, we obtain Lagrange interpolation polynomials, which cannot be used for the uniform approximation of continuous functions. In his work, Stancu considered only non-negative values of the replacement parameter in the Pólya urn model.

In [48], [49], and [70], the operators R_n were introduced (4.8). It was shown that by choosing negative values for the replacement parameter in the Pólya urn model, the operators R_n provide better approximation results than all Bernstein-Stancu type operators.

This chapter provides an extension to the previous results and shows that, for a sufficiently large class of functions, namely convex functions, and for a minimal admissible choice of the replacement parameter, the operators R_n offer the best approximation of convex functions.

The proof of the main result is given by Theorem 4.2.1 and is based on an intermediate result, given by Theorem 4.1.1, which shows that Pólya random variables satisfy convex ordering with respect to the replacement parameter. In turn, the proof of this result relies on two other results: a result regarding the ordering (interlacing) of three sets (Lemma 4.1.1) and the monotonicity of the first centered moment of the Pólya distribution (Lemma 4.1.2).

4.1 Preliminary results on operators constructed with the Pólya distribution

We recall that the integer parameters $a, b, c \geq 0$ and $n \geq 1$ must satisfy the compatibility condition:

$$a + (n-1)c \geq 0 \quad \text{and} \quad b + (n-1)c \geq 0. \tag{4.1}$$

The Pólya urn model involves determining the number of white balls (successes) drawn in n trials from an urn initially containing a white balls and b black balls, where after each draw, the drawn ball is replaced in the urn with c balls of the same color.

Let $X_n^{a,b,c}$ denote a Pólya random variable with parameters a, b, c, and n, which represents the number of successes in this experiment.

We have:

$$p_{n,k}^{a,b,c} = P\left(X_n^{a,b,c} = k\right) = C_n^k \frac{a^{(k,c)} b^{(n-k,c)}}{(a+b)^{(n,c)}}, \qquad k \in \{0, 1, \ldots, n\}, \tag{4.2}$$

where $x, h \in \mathbb{R}$, $n \in \mathbb{N}$, and

$$x^{(n,h)} = x\,(x+h)\,(x+2h)\cdots(x+(n-1)h) \tag{4.3}$$

By convention, $x^{(0,h)} = 1, \quad \forall x, h \in \mathbb{R}$.

It is known that (see [33]) the first centered moment of the Pólya distribution $X_n^{x,1-x,c}$ is given by:

$$E\left(\left(nx - X_n^{x,1-x,c}\right) 1_{X_n^{x,1-x,c} \leq s-1}\right) = s p_{n,s}^{x,1-x,c}\,(1 - x + (n-s)\,c) \tag{4.4}$$

for all $s \in \{1, \ldots, n\}$.

Let:

$$P_n^{a,b,c} : F\left([0,1]\right) \to F\left([0,1]\right),$$

denote the operators defined by:

$$P_n^{a,b,c}\,(f; x) = E\left(\frac{1}{n} X_n^{a,b,c}\right) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{a,b,c}, \tag{4.5}$$

where the parameters $a, b \geq 0$ and $c \geq -\frac{\min\{x,1-x\}}{n-1}$ may depend on $n \in \mathbb{N}^*$ and $x \in [0,1]$, and satisfy the compatibility condition (4.1) (see [48]).

For particular choices, we obtain Bernstein operators:

$$B_n(f;x) = P_n^{x,1-x,0}(f;x), \tag{4.6}$$

Bernstein-Stancu operators:

$$P_n^c(f;x) = P_n^{x,1-x,c}(f;x) \tag{4.7}$$

and the operators R_n introduced in [48]:

$$R_n(f;x) = P_n^{x,1-x,-\min\{x,1-x\}/(n-1)}, \tag{4.8}$$

which correspond to the minimal value of the replacement parameter c when $a = x$ and $b = 1 - x$. The compatibility condition (4.1) is satisfied.

We recall that a random variable X is said to be smaller in convex order than a random variable Y and we denote $X \leq_{\text{cx}} Y$ if and only if:

$$E(\phi(X)) \leq E(\phi(Y)) \tag{4.9}$$

for any convex function $\phi : \mathbb{R} \to \mathbb{R}$ for which the above expectations exist. If X and Y are random variables for which the means $E[X]$ and $E[Y]$ exist and are equal, it is known (see [59], Theorem 3.A.1) that:

$$E((t-X)_+) \leq E((t-Y)_+), \qquad \forall t \in \mathbb{R}, \tag{4.10}$$

where $x_+ = \max\{x, 0\}$.

Lemma 4.1.1. *[44] For any $x \in (0,1)$, integers $n \geq 3$, and $k \in \{1, \ldots, n-2\}$, there exists a partition $\{n_1, \ldots, n_k\} \bigsqcup \{m_1, \ldots, m_{n-k-1}\}$ of the set $\{1, 2, \ldots, n-1\}$ such that:*

$$n_i \leq \frac{i}{x}, \quad i \in \{1, \ldots, k\} \qquad \textit{and} \qquad m_i \leq \frac{i}{1-x}, \quad i \in \{1, \ldots, n-k-1\}. \tag{4.11}$$

Lemma 4.1.2. *[44] For any $x \in [0,1]$ and any integers $n \geq 1$ and $k \in \{0, 1, \ldots, n\}$, the sum*

$$\sum_{i=0}^{k} \left(x - \frac{i}{n}\right) p_{n,i}^{x,1-x,c} \tag{4.12}$$

is a non-decreasing function in $c \geq -\frac{1}{n-1}\min\{x, 1-x\}$.

Furthermore, for $x \in (0,1)$ and any integers $n \geq 2$ and $k \in \{0, \ldots, n-1\}$, the above sum is increasing with respect to $c \geq -\frac{1}{n-1}\min\{x, 1-x\}$.

Theorem 4.1.1. *[44] The Polya random variable $X_n^{x,1-x,c}$ satisfies the following convex ordering:*

$$X_n^{x,1-x,c} \leq_{cx} X_n^{x,1-x,c'}, \tag{4.13}$$

for any integer $n \geq 1$, $x \in [0,1]$, and any $c' \geq c \geq -\frac{1}{n-1}\min\{x, 1-x\}$.

4.2 Error Estimation of Polya-Stancu Operators

As an application of Theorem 4.1.1, we have:

Theorem 4.2.1. *[44] For any convex function $f : [0,1] \to \mathbb{R}$, the approximation error of the Polya-Stancu operators P_n^c is a non-decreasing function with respect to c, such that:*

$$\left|P_n^{c_2} f(x) - f(x)\right| \geq \left|P_n^{c_1} f(x) - f(x)\right|, \tag{4.14}$$

for any integer $n \geq 2$, $x \in [0,1]$, and any $c_2 > c_1 \geq -\frac{1}{n-1}\min\{x, 1-x\}$.

Furthermore, if

$$B_n f(x) \neq B_1 f(x) \tag{4.15}$$

for some values of $n \geq 2$ and $x \in [0,1]$, then the above inequality is strict, hence:

$$\left|P_n^{c_2} f(x) - f(x)\right| > \left|P_n^{c_1} f(x) - f(x)\right|, \tag{4.16}$$

for any $c_2 > c_1 \geq -\frac{1}{n-1}\min\{x, 1-x\}$.

Chapter 5

Summation methods applied to operators

This chapter contains results presented by the author in the paper [42], in collaboration with Radu Păltănea.

If we consider the inner product induced by a weight on the space of functions integrable with respect to this weight and an orthonormal family in this space, then each function in this space can be associated with a generalized Fourier series. It is well known that, generally, the sequence of partial sums of the Fourier series associated with a function in this space does not converge uniformly to the given function over the interval (Lozinsky and Harshiladze's Theorem, [57]), although they may converge pointwise, as well as in the norm induced by the inner product. A classic example of this is the sequence of partial sums of trigonometric Fourier series (the Gibbs phenomenon).

In the case of trigonometric Fourier series, Fejér demonstrated that when the sequence of operators is constructed as the Cesàro mean of the partial sums of the Fourier series, known as Fejér's operators, these operators are linear and positive and have the property of uniform approximation of continuous functions.

More recently, it has been shown that linear and positive operators of Durrmeyer [24], or more generally, Durrmeyer operators with Jacobi weights [50], which have the property of uniform approximation of continuous functions on the interval $[0, 1]$, can be represented as modified partial sums of the Fourier series of Legendre and Jacobi polynomials, respectively. This chapter highlights that both Durrmeyer operators and Durrmeyer operators with Jacobi weights can be constructed by applying regular summation methods to the partial sums of the generalized Fourier expansion of orthogonal polynomials.

The chapter is organized as follows: the initial sections present generalities regarding trigonometric Fourier series and Fejér's operators, generalized Fourier series, projectors, and summation methods. The final section contains the core results, which constitute the original contribution of this chapter.

5.1 Generalities on Summation Methods

Consider the set of all real sequences:

$$\mathcal{S} = \{\sigma \mid \sigma = (s_n)_{n\geq 0}\}$$

on which summation and multiplication by a real number can be defined. Due to this, $\mathcal{S}$ can be viewed as a vector space.

A sequence $(s_n)_{n\geq 0}$ will be denoted by $(s_n)_n$ or, more simply, as (s_n).

Definition 5.1.1. *A transformation is defined as a mapping T on a subset $T(F)$ of $\mathcal{S}$ that maps an element $\sigma \in \mathcal{F}(\mathcal{T})$ to an element in $\mathcal{S}$, denoted by $T\sigma$.*

Definition 5.1.2. *Let $\mathcal{F}(\mathcal{T}) = \{\sigma \in \mathcal{S} \mid \mathcal{T}\sigma \text{ exists}\}$. A transformation T is called linear if the following relations are true:*

$$T(\sigma + \tau) = T\sigma + T\tau, \quad T(\alpha \cdot \sigma) = \alpha \cdot T\sigma \quad \textit{for } \sigma, \tau \in \mathcal{S},\ \alpha \in \mathbb{R}.$$

We introduce the following notations:

$$\begin{aligned} \mathcal{S}_c &= \{\sigma \in \mathcal{S} \mid \sigma \text{ converges}\}, \\ \mathcal{F}(\mathcal{T}) &= \{\sigma \in \mathcal{F}(\mathcal{T}) \mid \mathcal{T}\sigma \in \mathcal{S}\}. \end{aligned}$$

Consider now the operator:

$$\lim : \mathcal{S}_c \to \mathbb{R},$$

which maps a convergent sequence to its limit. We also denote by:

$$T - \lim : F_c(T) \to \mathbb{R},$$

the generalized operator that maps the limit of a sequence in the space $F_c(T)$, defined by:

$$T - \lim = \lim \circ T.$$

Definition 5.1.3. *With the above notations and (s_n), we have:*

$$T - \lim((s_n)) = \lim_{n\to\infty} (t_n), \tag{5.1}$$

where $(t_n) = T((s_n))$. If it exists, $T - \lim$ is called the generalized limit induced by the transformation T.

Definition 5.1.4. *A transformation is called regular if it transforms convergent sequences into convergent sequences with the same limit.*

Next, we consider transformations generated by matrices, known as matrix transformations.

Let A be the matrix:

$$A = (a_{nm}) = \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0n} & \dots \\ a_{10} & a_{11} & \dots & a_{1n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n0} & a_{n1} & \dots & a_{nm} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Consider the transformation $T_A : \mathcal{F}(\mathcal{T}) \to \mathcal{S}$ which associates a sequence (s_n) with (t_n) defined by:

$$t_n = \sum_{m=0}^{\infty} a_{nm} s_m, \quad n \geq 0, \tag{5.2}$$

where $\mathcal{F}(\mathcal{A})$ consists of those sequences (s_n) for which the series above converges for every n.

Definition 5.1.5. *Let $A = (a_{nm})$. The transformation T_A is called regular if A is a regular matrix.*

Theorem 5.1.1. *(Petersen, [54]) Let $A = (a_{nm})$. We have:*
a) $(T_A)\sigma$ exists for any bounded sequence σ if and only if:

$$\sum_{m=0}^{\infty} |a_{nm}| \text{ converges for every } n.$$

b) $(T_A)\sigma$ exists for any bounded sequence σ if and only if $(T_A)\sigma$ exists for any sequence σ that converges to 0.

Theorem 5.1.2. *(Petersen, [54]) A necessary and sufficient condition for the matrix $A = (a_{nm})$ to transform every bounded sequence into a bounded sequence is that there exists k such that:*

$$\sum_{m=0}^{\infty} |a_{nm}| \leq k \text{ for every } n.$$

Next, we address the conditions satisfied by a regular matrix.

Theorem 5.1.3. *(Petersen, [54]) [Schur's Theorem] The matrix (a_{nm}) is regular if and only if:*

(i) There exists K such that:

$$\sum_{m=0}^{\infty} |a_{nm}| < K, \quad \forall n;$$

(ii) $\forall m, \lim_{n\to\infty} a_{nm} = 0$;

(iii) $\lim_{n\to\infty}\sum_{m=0}^{\infty} a_{nm} = 1$.

Definition 5.1.6. *A matrix is called a Schur matrix if it transforms every bounded sequence into a convergent sequence.*

Theorem 5.1.4. *(Petersen, [54]) For a matrix $A = (a_{nm})$ to be a Schur matrix, the following conditions are necessary and sufficient:*

(i) $\lim_{n\to\infty} a_{nm}$ *exists for all* m;

(ii) $\sum_{m=0}^{\infty} |a_{nm}|$ *converges for all* n *and the convergence is uniform with respect to* n.

Corollary 5.1.1. *If $A = (a_{nm})$ is regular, there exists a bounded sequence that is transformed by A into a divergent sequence.*

Corollary 5.1.2. *The matrix $A = (a_{nm})$ makes all bounded sequences converge to* 0 *if and only if:*

$$\sum_{m=0}^{\infty} |a_{nm}| \text{ converges for all } n \text{ and}$$

$$\lim_{n\to\infty}\sum_{m=0}^{\infty} |a_{nm}| = 0. \tag{5.3}$$

An important class of matrices is the class of Toeplitz matrices.

Definition 5.1.7. *A matrix $A = (a_{nm})_{n,m\geq 0}$ is called a Toeplitz matrix if it is lower triangular, i.e., it has the property that $a_{nm} = 0$ whenever $m > n$.*

In the case of Toeplitz matrices, Theorem 5.1.1 is no longer necessary, as for any Toeplitz matrix A and for any sequence σ, $(T_A)\sigma$ exists.

Additionally, for Toeplitz matrices, the generalized limit of sequences is written more simply as:

$$A - \lim_{n\to\infty} s_n := \lim_{n\to\infty} t_n, \tag{5.4}$$

where

$$t_n = \sum_{m=0}^{n} a_{n,m} s_m,\ n \geq 0. \tag{5.5}$$

Definition 5.1.8. *Two matrix methods A and B are said to be consistent if, for any sequence (s_n) that is convergent by both methods, the limits are equal:*

$$A - \lim s_n = B - \lim s_n.$$

Definition 5.1.9. *A family $\mathfrak{T}$ of methods that transforms bounded sequences into bounded sequences is called consistent if any two methods in $\mathfrak{T}$ are consistent.*

Definition 5.1.10. *Given two methods that transform bounded sequences into bounded sequences, T_1 and T_2, if every T_2-convergent sequence is convergent with and by the transformation T_1 with the same limit, we say that T_1 is stronger than T_2 and denote it as $T_1 \supseteq T_2$.*

Definition 5.1.11. *Let A and B be two matrices. We say that A is stronger than B if the transformation T_A is stronger than the transformation T_B.*

Theorem 5.1.5. *(Petersen, [54]) The method T_A is stronger than the method T_B if and only if the method $T_{A\cdot B^{-1}}$ is regular.*

Definition 5.1.12. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two families of methods that transform bounded sequences into bounded sequences. We say that the family $\mathcal{T}_1$ is stronger than the family $\mathcal{T}_2$ if for any sequence (s_n) that is made convergent by any of the methods $T_2 \in \mathcal{T}_2$, there exists a method $T_1 \in \mathcal{T}_1$ that transforms the sequence (s_n) into a convergent sequence such that:*

$$T_1 - \lim s_n = T_2 - \lim s_n.$$

We then write:

$$\mathcal{T}_1 \supset \mathcal{T}_2.$$

Definition 5.1.13. *The family of matrices $\mathcal{A}$ is stronger than the family of matrices $\mathcal{B}$ if the family of methods associated with $\mathcal{A}$, $\mathcal{T}_{\mathcal{A}}$, is stronger than the family of methods associated with $\mathcal{B}$, $\mathcal{T}_{\mathcal{B}}$.*

We say that the method given by:

$$t_n = \frac{s_0 + \ldots + s_n}{n+1}$$

is called the Cesàro method of first order.

Theorem 5.1.6. *(Petersen, [54]) The Cesàro method is a regular method.*

For the higher-order Cesàro method, consider the sequence $(s_n)_{n\geq 0}$. We define the higher-order sums as follows:

$$s_n^1 = s_0 + \cdots + s_n, \quad n \geq 0, \tag{5.6}$$

$$s_n^{k+1} = s_0^k + \cdots + s_n^k, \quad n \geq 0,\, k \geq 1.$$

We can write:

$$s_n^k = \sum_{j=0}^{n} \sigma_{nj}^k s_j, \quad n \geq 0,\, k \geq 1.$$

The following formula is proved by induction on k:

$$\sigma_{nj}^k = \binom{n+k-j-1}{k-1}. \tag{5.7}$$

The Cesàro method of k- th order is defined by the lower triangular matrix:

$$C^k = (c_{nm}^k)_{n,m\geq 0}, \quad k \geq 1, \tag{5.8}$$

where

$$c_{nm}^k = \begin{cases} \dfrac{\sigma_{nm}^k}{\sum\limits_{j=0}^{n} \sigma_{nj}^k} & \text{for } 0 \le m \le n, \\ 0 & \text{for } m > n. \end{cases} \tag{5.9}$$

We have:

$$\begin{aligned} \sum_{j=0}^{n} \sigma_{nj}^k &= \sum_{j=0}^{n} \binom{n+k-j-1}{k-1} \\ &= \sum_{j=0}^{n} \left[\binom{n+k-j}{k} - \binom{n+k-j-1}{k} \right] \\ &= \binom{n+k}{k} - \binom{k-1}{k} \\ &= \binom{n+k}{k}. \end{aligned}$$

Thus,

$$C^k = (c_{nm}^k)_{n,m}, \quad c_{nm}^k = \binom{n+k}{k}^{-1} \binom{n+k-m-1}{k-1}. \tag{5.10}$$

Therefore, the transformation of the sequence (s_n) is the sequence (t_n) defined as:

$$t_n = \sum_{m=0}^{n} c_{nm}^k s_m.$$

Theorem 5.1.7. *(Petersen, [54]) The Cesàro matrix of order k is regular.*

Theorem 5.1.8. *(Petersen, [54]) If $k_1 \le k_2$, then the summation method (C, k_2) is stronger than the summation method (C, k_1).*

5.2 Modified Fourier-Jacobi Partial Sums

We will show that for the Fourier-Legendre and Fourier-Jacobi partial sums, transformations with Toeplitz convergence matrices can be applied in such a way that the resulting operators possess the property of uniform approximation of all continuous functions. These modifications correct the negative convergence result in the theorem of Lozinsky and Harshiladze.

These modifications are analogous to the modification of trigonometric Fourier series that leads to Fejér's operators, obtained by applying the Cesàro summation method to the partial sums of the Fourier series.

On the interval $[0,1]$, consider the Jacobi weight $\rho(t) = t^a(1-t)^b$, where $a > -1$ and $b > -1$. This weight introduces an inner product on $C([0,1])$ given by:

$$< f,g >_{a,b}= \int_0^1 f(t)g(t)t^a(1-t)^b\,dt, \quad f,g \in C([0,1]). \tag{5.11}$$

Let $p_n^{a,b}$ denote the Jacobi orthogonal polynomials of degree n associated with the weight $\rho(t)$, which satisfy the equations $\langle p_n^{a,b}, p_m^{a,b}\rangle_{a,b} = \delta_{n,m}$.

The formal Fourier-Jacobi series associated with a function $f \in C([0,1])$ is given by:

$$f(x) \sim \sum_{n=0}^{\infty} \langle f, p_n^{a,b}\rangle_{a,b} \cdot p_n^{a,b}(x), \quad x \in [0,1]. \tag{5.12}$$

By a simple translation, the analogous Fourier-Jacobi series for a function $f \in C([-1,1])$ can be constructed by replacing the interval $[0,1]$ with $[-1,1]$ and the weight ρ with $\rho(x) = (1-x)^a(1+x)^b$. The properties of the two series are similar.

Since the partial sums of degree n of the series (5.12) are projectors onto the polynomial spaces Π_n, by the Lozinsky-Harshiladze theorem, the series (5.11) does not converge uniformly to f on the interval $[0,1]$ for any function $f \in C([0,1])$.

To extend the domain of convergence of the series (5.12), we can apply generalized summation methods. If $A = (a_{n,m})_{n,m}$ is a Toeplitz matrix, then we apply the transformation given by the matrix A to the partial sums of the series (5.12). Specifically, if we denote the sequence of partial sums by (s_n), we consider its generalized limit according to the formulas (5.4), (5.5). The summation method $(C,1)$ applied to the sequence of partial sums of the Fourier series leads to Fejér's operators, which provide uniform approximation properties for all continuous and periodic functions, whereas the sequence of partial sums of the Fourier series only has pointwise approximation properties.

In the case of Jacobi series, summation methods have been applied only to achieve pointwise convergence at the endpoints of the interval. For instance, Szegő ([65]) provides the following result:

Theorem 5.2.1. *(Szegő [65]) Let $f(x)$ be a continuous function on $[-1,1]$. The expansion of $f(x)$ in a Jacobi series with weight $\rho(x) = (1-x)^a(1+x)^b$, where $a,b > -1$, $x \in [-1,1]$, is (C,k)-summable at $x = 1$ if $k > a + \frac{1}{2}$. In general, this is not true if $k = a + \frac{1}{2}$. Analogously, this holds for $x = -1$, with a replaced by b.*

More recently, the Nörlund summability problem for the Jacobi series of a function $f \in C([-1,1])$ but still only pointwise at the endpoints of the interval $[-1,1]$ has been studied in: Gupta ([30]), Thorpe ([67]), Singh ([58]), and Chandra ([19]). In this section, we highlight that there exists a summation method which allows for the uniform summation of the Jacobi-Fourier series for all continuous functions. The method relies on reducing the problem of summing the Jacobi-Fourier partial sums to the problem of uniform convergence of the sequence of Durrmeyer operators with Jacobi weight. The main result is to demonstrate that this method is stronger than all Cesàro methods of any order.

We use the convention that $\binom{p}{r} = 0$ if $p, r \in \mathbb{Z}$ and the condition $0 \le r \le p$ is not satisfied. We denote $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The Durrmeyer operators with Jacobi weight $\rho(t) = t^a(1-t)^b$, where $a > -1$ and $b > -1$, were first introduced in [50] and then studied in many other works (see [51], [12], or [53]), and are defined as follows:

$$\begin{aligned} M_n^{a,b}(f)(x) &= \sum_{k=0}^{n} \frac{< p_{n,k}, f >_{a,b}}{< p_{n,k}, e_0 >_{a,b}} p_{n,k}(x), \\ &= (a+b+n+1) \sum_{k=0}^{n} \left(\int_0^1 f(t) p_{n,k}(t) t^a (1-t)^b \, dt \right) p_{n,k}(x), \end{aligned}$$

where $f \in L([0,1])$, $x \in [0,1]$, $n \in \mathbb{N}$, $a > -1$, $b > -1$, and $\langle \varphi, \psi \rangle_{a,b}$, $\varphi, \psi \in L([0,1])$.

In the special case $a = 0$ and $b = 0$, these operators become the Durrmeyer operators introduced in [25] and independently in [39].

The operators $M_n^{a,b}$ are linear and positive. A fundamental property of these operators is given by the following theorem.

Theorem 5.2.2. *(Păltănea, [50]) For any $f \in C([0,1])$, we have:*

$$\lim_{n\to\infty} M_n^{a,b}(f)(x) = f(x), \text{ uniformly for } x \in [0,1]. \tag{5.13}$$

The operators $M_n^{a,b}(f)(x)$ can also be represented as modified Fourier partial sums with respect to the scalar product defined earlier.

This was discovered by Dierrennic in the case $a = 0$, $b = 0$, in the paper [24] and was extended to the general case $a > -1$, $b > -1$ in the paper [50]. This representation fully describes the spectral structure of these operators. We have:

Theorem 5.2.3. *(Păltănea, [50]) The operators $M_n^{a,b}$ admit a representation in the form:*

$$M_n^{a,b}(f)(x) = \sum_{m=0}^{n} \lambda_{n,m}^{a,b} \langle f, p_m^{a,b} \rangle_{a,b} p_m^{a,b}(x), \; f \in L([0,1]), \; x \in [0,1], \; n \in \mathbb{N}, \tag{5.14}$$

where $p_m^{a,b}$ are the Jacobi polynomials normalized by the condition $\langle p_m^{a,b}, p_m^{a,b} \rangle_{a,b} = 1$ and

$$\lambda_{n,m}^{a,b} = \frac{\Gamma(n+1)\Gamma(n+2+a+b)}{\Gamma(n-m+1)\Gamma(n+m+2+a+b)}, \; 0 \le m \le n, \tag{5.15}$$

where Γ is the Gamma function.

For $a > -1$, $b > -1$, we consider the infinite matrix $\Gamma^{a,b} = (\gamma_{n,m}^{a,b})_{n,m}$, where

$$\gamma_{n,m}^{a,b} = \begin{cases} \frac{n!\Gamma(n+2+a+b)(2m+2+a+b)}{(n-m)!\Gamma(n+m+3+a+b)}, & 0 \le m \le n \\ 0, & m > n \ge 0 \end{cases} \tag{5.16}$$

We denote by $\Gamma^{a,b} - \lim_{n\to\infty} a_n$ the generalized limit of the sequence $(a_n)_n$ with respect to the method given by $\Gamma^{a,b}$, and by $\Gamma^{a,b} - \sum_{n=0}^{\infty} a_n$ the generalized sum of the series $\sum_{n=0}^{\infty} a_n$ with the summation method given by $\Gamma^{a,b}$.

The fundamental result we wish to highlight is that the representation of the operators $M_n^{a,b}$ given in formulas (5.14) and (5.15) can be interpreted as the result of applying the transformation given by the matrix $\Gamma^{a,b}$ to the sequence of partial sums of the series (5.12). Thus, we have:

Theorem 5.2.4. *[42] The sequence of operators $(M_n^{a,b})_{n\geq 0}$ can be written in the form:*

$$M_n^{a,b}(f)(x) = \sum_{m=0}^{n} \gamma_{n,m}^{a,b} S_m^{a,b}(f)(x),\ f \in L([0,1]),\ x \in [0,1],\ n \in \mathbb{N}, \tag{5.17}$$

where

$$S_m^{a,b}(f)(x) = \sum_{k=0}^{m} \langle f, p_k^{a,b}\rangle_{a,b} p_k^{a,b}(x). \tag{5.18}$$

The first main result is the following theorem:

Theorem 5.2.5. *[42] For any $f \in C([0,1])$, $a > -1$, $b > -1$, we have:*

$$\Gamma^{a,b} - \sum_{n=0}^{\infty} \langle f, p_k^{a,b}\rangle_{a,b} p_k^{a,b}(x) = f(x),\ x \in [0,1] \tag{5.19}$$

and the convergence of the series is uniform in x.

Remark 5.2.1. From the previous theorem, we conclude that the regular matrix $\Gamma^{a,b}$ performs the summation of the series (5.12).

5.3 Comparison of the new summation method with Cesàro methods

Lemma 5.3.1. *[42] Let $k \in \mathbb{N}$. The inverse of the matrix C^k defined in (5.10) is the matrix $D^k := (C^k)^{-1}$, where $D^k = (d_{n,m}^k)_{n,m}$ and*

$$d_{n,m}^k = \begin{cases} (-1)^{n+m}\binom{m+k}{k}\binom{k}{n-m}, & n-k \leq m \leq n, \\ 0, & 0 \leq m < n-k \text{ or } m > n. \end{cases} \tag{5.20}$$

We define

$$\alpha_{n,m}^c := \frac{n!\Gamma(n+c)}{(n-m)!\Gamma(n+m+c)}, \quad c \geq 0,\ n \in \mathbb{N},\ m \in \mathbb{N}_0,\ 0 \leq m \leq n. \tag{5.21}$$

We observe that

$$\lambda_{n,m}^{a,b} = \alpha_{n,m}^c,\ \text{for } 0 \leq m \leq n,\ \text{if } c = a+b+2,\ a > -1,\ b > -1. \tag{5.22}$$

Also,

$$\alpha^c_{n,m} \le \alpha^0_{n,m} = \frac{n!(n-1)!}{(n-m)!(n+m-1)!},\ 0 \le m \le n,\ n \ge 1,\ c \ge 0. \tag{5.23}$$

Indeed, we have:

$$\frac{\alpha^c_{n,m}}{\alpha^0_{n,m}} = \frac{\Gamma(n+c)\Gamma(n+m)}{\Gamma(n)\Gamma(n+m+c)} = \prod_{j=0}^{m-1} \frac{n+j}{n+c+j} \le 1.$$

For $c \ge 0$, $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \le m \le n-p$, we define:

$$U^{c,p}_{n,m} = \sum_{i=0}^{p} (-1)^i \binom{p}{i} \alpha^c_{n,m+i}. \tag{5.24}$$

Lemma 5.3.2. *[42] Let $p \in \mathbb{N}_0$ and $c \ge 0$.*

i) There exist real coefficients $d_{2p,i}$, $0 \le i \le p$, such that for any $n \in \mathbb{N}$ and any $m \in \mathbb{N}_0$, $0 \le m \le n-2p$, the following holds:

$$|U^{c,2p}_{n,m}| \le \sum_{i=0}^{p} d_{2p,i} \frac{(m+c+2p)^{2i}}{n^{p+i}} \cdot \alpha^0_{n,m}. \tag{5.25}$$

ii) There exist real coefficients $d_{2p+1,i}$, $0 \le i \le p$, such that for any $n \in \mathbb{N}$ and any $m \in \mathbb{N}_0$, $0 \le m \le n-2p-1$, the following holds:

$$|U^{c,2p+1}_{n,m}| \le \sum_{i=0}^{p} d_{2p+1,i} \frac{(m+c+2p+1)^{2i+1}}{n^{p+1+i}} \cdot \alpha^0_{n,m}. \tag{5.26}$$

Corollary 5.3.1. *[42] For any $p \in \mathbb{N}$ and $c \geq 0$, we have:*

$$|U^{c,p}_{n,m}| \leq C^1_{c,p} \max\left\{ \left(\frac{1}{\sqrt{n}}\right)^p, \left(\frac{m+k}{n}\right)^p \right\} \alpha^0_{n,m}, \; 0 \leq m \leq n, \tag{5.27}$$

where

$$C^1_{c,p} = \left(\max\left\{1, \frac{c+p}{k}\right\}\right)^p \sum_{i=0}^{\left[\frac{p}{2}\right]} d_{p,i}. \tag{5.28}$$

Lemma 5.3.3. *[42] For real numbers $k \geq 1$ and $p \geq 1$, there exists a constant $C^2_{k,p}$ such that*

$$E_{k,p}(n,m) := \left(\frac{(m+k)^2}{n}\right)^p \alpha^0_{n,m} \leq C^2_{k,p}, \tag{5.29}$$

for all integers $0 \leq m \leq n$.

Corollary 5.3.2. *[42]*

For any $k \in \mathbb{N}$ and any real number $c \geq 0$, there exists a constant $C^3_{k,c}$ such that:

$$\binom{m+k}{k} |U^{c,k+1}_{n,m}| \leq C^3_{c,k}, \text{ for all integers } 0 \leq m \leq n. \tag{5.30}$$

Lemma 5.3.4. *[42] For any integer $k \geq 1$, $0 \leq m \leq n-k+1$, and real numbers $a > -1$, $b > -1$, let $c = a+b+2$. We have:*

$$\sum_{\mu=m}^{m+k-1} \sum_{i=0}^{m+k-1-\mu} (-1)^i \binom{k}{i} \binom{\mu+k}{k} \gamma^{a,b}_{n,\mu+i} = \sum_{j=0}^{k-1} \binom{m+k}{k-j} U^{c,k-j}_{n,m+j}. \tag{5.31}$$

Corollary 5.3.3. *[42] For any integer $k \geq 1$ and any numbers $a > -1$, $b > -1$, let $c = a+b+2$. There exists a constant $C^4_{c,k}$ such that:*

$$\left| \sum_{\mu=m}^{m+k-1} \sum_{i=0}^{m+k-1-\mu} (-1)^i \binom{k}{i} \binom{\mu+k}{k} \gamma^{a,b}_{n,\mu+i} \right| \leq C^4_{c,k}, \tag{5.32}$$

for any integers m and n such that $0 \leq m \leq n-k+1$.

The central result of this section is given by the following theorem.

Theorem 5.3.1. *[42] The summation method given by the matrix $\Gamma^{a,b}$, with $a > -1$ and $b > -1$, is more powerful than all Cesàro methods (C,k), where $k \in \mathbb{N}$ and $k \geq 1$.*

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