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# TEZĂ DE DOCTORAT

# Contributions to approximation by positive linear operators

# Abstract

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***TEZĂ DE DOCTORAT***

## Contribuții la aproximarea prin operatori liniari și pozitivi

## Contributions to approximation by linear positive operators

**Domeniul de doctorat: Matematică**

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# Cuprins

# Preface

The main theme of the thesis is to investigate current issues of approximation theory by linear and positive operators. The approximation theory is a vast field of mathematical analysis and focuses on the topics related ti the approximation of functions through other functions, which are simpler and easier to calculate.

The foundations of this theory appeared with Chebyshev's classical theorem of best approximation and Weierstrass's theorem of polynomial approximation of continuous functions from the late 19th century. The modern theory of approximation of continuous functions by linear and positive operators was consolidated with the theorem of Popoviciu (1950) -Bohman (1952) -Korovkin (1953).

Representative results from this theory were obtained by G. G. Lorentz, P. L. Butzer, R. DeVore, B. Sendov and V. Popov, Z. Ditzian, V. Totik, D. D. Stancu, A. Lupas, F. Altomare, M. Campiti and many others. The central themes of the approximation theory are: various methods of obtaining new operators, estimating the convergence rate of operators using continuity modules and K - Functional, or by asymptotic formulas, study the rest and its representation, global smoothness and conservation of certain properties, the most important being maintaining the convexity.

The main objective of the thesis is to obtain estimates, with optimal constants, of the approximation order using continuity modules (simple and weighted) and the moments of the operators. The paper is structured in four chapters, the first part of each chapter presents the basic notions, respectively notations used in it.

The paper begins by presenting the normed spaces of functions, the definition of linear and positive operators, the definition of moduli of continuity and weighted moduli of continuity, and also the presentation of their properties.

The first chapter begins by presenting the basics of approximation of functions by linear and positive operators on compact intervals, namely: convergence theorems for linear and positive operators, general evaluations using 1st and 2nd order moduli of continuity and Voronoskaja type theorems.

The own results consist in obtaining general asymptotic estimates using moduli of continuity and demonstrating the optimality of the constants that appear in these estimates.

The second chapter begins by introducing Bernstein's well-known operator along with a number of its fundamental properties. Bernstein polynomials helped to obtain a constructive procedure for proving Weierstrass's approximation theorem. For this operator, evaluations of the degree of approximation were obtained using moduli of continuity, the asymptotic constants obtained being optimal.

Also in this chapter other Bernstein type operators were presented together with their properties (Kantorovich operators, Durrmeyer operators, Durrmeyer-genuine operators, $\alpha$–Bernstein operators). An own result is an iterative method by which it was possible to obtain a new class of Bernstein-type operators, for which several properties were studied.

Chapter 3 has as its central theme the approximation of functions by linear and positive operators on non-compact, using two types of approximation: uniform approximation on compact intervals and weighted approximation. The own results consist in general estimates of the weighted approximation on non-compact intervals using classical and weighted moduli of continuity. Special attention is given to the weights 1 and $\frac{1}{x^2+1}$. The results obtained were applied to the Szász - Mirakijan and Baskakov operators.

At the end of the chapter, a Chlodovsky-type change is presented for the *alpha*–Bernstein operators. This is a method for obtaining new operators for non-compact approximation.

Chapter 4 presents general notions regarding the semigroups of operators. The first section contains definitions and basic results of the approximation of the semigroups of operators. Here, too, several quantitative variants of Trotter's theorem have been presented. The own results obtained consist in quantitative estimates for the limit of the semigroups of linear and positive operators, these being applied in the case of Durrmeyer type operators. The results obtained for the one-dimensional case of the semigroups generated by the Bernstein operator were extended and for the multidimensional case of the semigroups generated by the multidimensional Bernstein operator.

The bibliography includes 129 papers that are cited during the thesis in case of the results taken over and presented in the paper.

In this paper are presented the original results of the author obtained during the elaboration of the thesis. They are found in the author's six articles, five published and one in progress, two of which have been published in the Results Mathematics [95] and Semigroup Forum [96], which have an SRI coefficient of over 0.5. The published papers received 6 citations in ISI journals.

The author has participated in the following international conferences: Romanian - German Seminar on Approximation Theory and Applications "RoGer 2017" (Oradea) and "RoGer 2019" (Cluj Napoca), and Mathematics and Computer Science "MACOS 2018" (Braşov). All results taken from the literature are accompanied by citations and have no evidence. In each chapter are indicated the works where the original results of the author were included. The original results form entirely the content of sections 1.2, 2.3, 3.2, 3.3.2, 4.4, 4.5 and part of sections 2.1.2, 3.3, 3.5.

In conclusion, I would like to express my deep gratitude and appreciation to Prof. Dr. Univ. Radu Păltănea and to Prof. Dr. Dr. h. C. Heiner Gonska, the scientific supervisors of this Thesis, for the trust given, for the advice and encouragement they offered me, for the time and patience they always had in the extremely discussions of benefits that I had from throughout my doctoral student activity.

# Preliminary

In the following, we will introduce the definitions and basic notions that we used in the study of approximation operators.

We start by introducing the notion of linear and positive operator as follows:

**Definition 0.0.1.** *An operator $L$ defined on a linear space of functions $V$ is named linear if*

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g),\ (\forall) f, g \in V\ si\ \alpha, \beta \in \mathbb{R}.$$

*The operator $L$ is positive if and only if*

$$L(f) \geq 0$$

*for all $f \in V$ and $f \geq 0$.*

In the following we will consider the following notations that we will use throughout the paper:

$$\begin{aligned}
&\mathbb{N} = \{1, 2, \ldots\},\ \mathbb{N}_0 = \mathbb{N} \cup \{0\} \\
&\Pi_n - \text{ the set of polynomials of degree less or equal than n, where } n \in \mathbb{N}_0 \\
&[a] - \text{ integer part of a} \\
&\{a\} - \text{the fractional part of a} \\
&e_n\ - \text{function } e_n(t) = t^n,\ t \in \mathbb{R},\ n \in \mathbb{N}_0
\end{aligned}$$

and the follow conventions:

$$\begin{aligned}
0^0 &= 1 \\
0 \cdot \infty &= 0 \\
\sum_{i=r}^{s} a_i &= 0,\ \prod_{i=r}^{s} a_i = 1 \text{ if } r > s \\
\binom{n}{k} &= 0 \text{ for } k \geq n
\end{aligned}$$

The restriction of a function $f : I \to \mathbb{R}$ to a set $J \subset I$ is noted with $f$.
We define on the interval $I$ the following spaces:

- $\mathcal{F}(I)$ - the space of real functions defined on $I$
- $\mathcal{F}_b(I)$ - the space of real functions locally bounded defined on $I$
- $B(I)$ - the space of real bounded functions defined on $I$
- $C(I)$ - the space of real continuous functions defined on $I$

- $C_b(I)$ - the space of real continuous and locally bounded functions define on $I$
- $\mathcal{D}(I)$ - the space of real continuous and differentiable functions define on $I$
- $\mathcal{L}_\mu(I)$ - the space of real integrable functions with measure $\mu$ defined on $I$

For every function $f \in B(I)$ we note the supremum norm as:

$$||f|| := \sup_{x\in I} |f(x)|$$

The test functions $e_i(x) = x^i,\ x \in I$.
The divided difference of function $f \in \mathcal{F}(I)$ in the nodes $x_1, \ldots, x_k$ is

$$[f; x_1, \ldots, x_k] := \sum_{i=1}^{k} \left( \prod_{1\le j\le k\ j\ne i} (x_i - x_j)^{-1} \right) f(x_i).$$

The finite difference of order k for the function $f \in \mathcal{F}(I)$ is

$$\Delta_h^k f(x) := \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x + jh),\ x \in I,\ x + kh \in I,\ h > 0$$

**Definition 0.0.2.** *[98] Function $f \in \mathcal{F}(I)$ is convex of order $k \ge -1$ if $[f; x_1, \ldots, x_{k+2}] \ge 0$ for each $x_1, \ldots x_{k+2}$ distinct on $I$.*

So $f$ is convex of order $-1$, if is positive, is convex of order 0, if is an increasing function and is convex of order 1 if is usually convex.

**Definition 0.0.3.** *Let be $V$ a linear space of $\mathcal{F}(I)$ and $k \ge -1$. The linear operator $L : V \to \mathcal{F}(I)$ is convex of order $k$ if for each function $f$ convex of order $k$, $L(f)$ is convex of order $k$.*

For each $L : V \to \mathcal{F}(I)$ linear and positive operator, $V \subset \mathcal{F}(I)$ we write $L(f)(x) = L(f; x)$. There are two main types of linear and positive operators:

1) Integral operators:

$$L : C(I) \to \mathcal{F}(I) \qquad L(f; x) := \int_I f(t)K(t, x)dt,\ f \in C(I),\ x \in I,$$

if $I$ is compact or integral is convergent, and $K \ge 0$ is a continuous global function.

2) Discrete operators:

$$L(f; x) = \sum_{i=1}^{\infty} f(\xi_i)\psi_i(x), \quad f \in \mathcal{F}(I),\ x \in I,\ \text{unde } \xi_i \in I,\ \psi_i \in \mathcal{F}(I),\ \psi_i \ge 0,$$

and the series is convergent. In a particular case we have the operators defined by finite sums.

**Definition 0.0.4.** *Let be $L : V \to \mathbb{R}$ a positive linear operator, where $V$ is a linear subspace of it $\mathcal{F}(I)$. Let be $x \in I$ fixed. The moments of the operator $L$ centered in $x$ are defined as:*

$$m_j(x) := L\left( (e_1 - xe_0)^j ; x \right), \quad j = 0, 1, \ldots$$

The degree of approximation by linear and positive operators depends on the smoothness properties of the functions. In approximations of the degree of approximation, instruments for measuring the smoothness of the functions represent the moduli of continuity.

**Definition 0.0.5.** *[5] Let be $f \in B(I)$, where $I$ is an interval. The application $\omega(f, \bullet) : [0, \infty) \to \mathbb{R}$,*

$$\omega(f; h) := \sup \{|f(x) - f(y)|,\ x, y \in I,\ |x - y| \leq h\rangle$$

*is named moduli of continuity or first order of $f$.*

**Theorem 0.0.1.** *[5](Properties of the first order moduli of continuity) Let be $f \in B(I)$. Then $\omega(f, \cdot)$ has the following properties:*

*1)* $\omega(f, \cdot) \geq 0$.

*2)* $\omega(f, 0) = 0$.

*3)* $\omega(f, \cdot)$ *is increasing function.*

*4)* $\omega(f, \cdot)$ *is subadditive.*

*5)* $\omega(f, \cdot)$ *is uniformly continuous, if* $f \in C_b(I) \cap B(I)$.

*6)* $(\forall)h \geq 0,\ (\forall)n \in \mathbb{N},\ \omega(f, nh) \leq n\omega(f, h)$.

*7)* $(\forall)h \geq 0, (\forall)\lambda > 0,\ \ \omega(f, \lambda h) \leq (1 + \lambda)\omega(f, h)$.

*8)* $(\forall)h \geq 0,\ (\forall)s \geq 1,\ |f(x) - f(y)| \leq (1 + h^{-s}|x - y|^s)\omega(f, h)$.

*9)* $(\forall)f_1,\ f_2 \in B(I),\ \omega(f_1 f_2; h) \leq ||f_1||\omega(f_2; h) + ||f_2||\omega(f_1; h)$.

**Definition 0.0.6.** *[87] Let $V \subset \mathcal{F}(I)$ be a linear subspace such that $\prod_k \subset V,\ k \in \mathbb{N}$. A function $\Omega_k : V \times [0, \infty) \to [0, \infty) \cup \{\infty\}$ is named moduli of continuity of order $k$ on $V$, if it verifies the following conditions:*

*1)* $\Omega_k(f, h_1) \leq \Omega_k(f, h_2),\ f \in V,\ \ 0 \leq h_1 < h_2$

*2)* $\Omega_k(f + p, h) = \Omega(f, h),\ \ f \in V,\ p \in \prod_{k-1},\ h > 0$

*3)* $\Omega_k(0, h) = 0,\ h > 0$

*We say that $\Omega_k$ is normed if it exists $M > 0$ such that $\Omega_k(e_k, h) \leq Mh^k,\ (\forall)h > 0$.*

The usually moduli of order $k \geq 1$ are defined as:

$$\omega_k(f, h) = \sup \left\{|\Delta_\rho^k f(x)|,\ x \in I, x + k\rho \in I,\ \rho \leq h\right\},$$

where $f \in B(I),\ h > 0$.
The moduli $\omega_1$ is simple note as $\omega$.

An admissible weighted on the interval $[0, 1]$ is a function $\varphi \in C[0, 1]$ such that $\varphi(x) > 0$, $x \in (0, 1)$.

The weighted moduli of first order of $f \in B[0, 1]$ is given as

$$\omega_2^\varphi(f, h) = \sup \left\{|f(u) - f(v)|,\ u, v \in [0, 1],\ |v - u| \leq h\varphi\left(\frac{u + v}{2}\right)\right\},$$

and the second order weighted modului of $f \in B[0, 1]$ is given by the following formula:

$$\omega_2^\varphi(f, h) = \sup \{|f(x - \rho) - 2f(x) + f(x + \rho)|,\ x \pm \rho \in [0, 1],\ |\rho| \leq h\varphi(x),\ h > 0\}$$

If $\varphi(x) = \sqrt{x(1 - x)},\ x \in [0, 1]$ we obtain the Ditzian–Totik moduli of first and second order and for $\varphi(x) = 1$ we obtain the classical moduli of continuity of order 1 and 2.

# Chapter 1

# Approximation operators for compact intervals

## 1.1 Approximation of functions by linear and positive operators

### 1.1.1 Convergence theorems

The classical theorem of approximation of continuous functions is the following:

**Theorem 1.1.1.** *[10][Weierstrass–1885] For each function $f(x)$ continuous on an interval $[a,b]$ and for each $\epsilon > 0$ it exists an polynomial $P(x)$ which approximates $f(x)$ uniformly with an error less than $\epsilon$:*

$$|f(x) - P(x)| < \epsilon,\ x \in [a,b]. \tag{1.1}$$

For the approximation of continuous functions, a basic tool is represented by linear and positive operators. The foundation of the theory of approximation by sequences of linear and positive operators was given by T. Popoviciu citePopoviciu2, H. Bohman [21] and P. P. Korovkin [76].

**Theorem 1.1.2** (T. Popoviciu–1950)**.** *[100] Let be a sequence of linear and positive operators as*

$$L_n(f;x) = \sum_{i=1}^{m_n} f\left(\xi_{n,i}\right)\psi_{n,i}(x),\ f \in C[a,b],\ x \in [a,b]$$

*where $\xi_{n,i} \in [a,b]$ and $\psi_{n,i}$ are positive polynomials. Suppose that*

$$L_n(e_0;x) = e_0$$

*and*

$$\lim_{n\to\infty} L_n\left((e_1 - xe_0)^2, x\right) = 0\ uniform\ in\ raport\ cu\ x \in [a,b]$$

*The we have*

$$\lim_{n\to\infty} L_n(f) = f\ uniform\ pe\ [a,b],\ \forall f \in C[a,b].$$

**Theorem 1.1.3** (P. P. Korovkin–1953)**.** *[76] Let be a sequence of linear and positive operators* $(L_n)_n,\ L_n : V \to \mathcal{F}[a,b]$*, where* $V$ *is a linear subspace of* $\mathcal{F}[a,b]$*. Suppose that* $\phi_0,\ \phi_1,\ \phi_2 \in V \cap C[a,b]$ *forms Cebisev's system on the interval* $[a,b]$*. If we have:*

$$\lim_{n\to\infty} L_n(\phi_i) = \phi_i \ uniformly\ for\ i = 0,1,2$$

*then*

$$\lim_{n\to\infty} L_n(f) = f \ uniformly\ for\ each\ f \in V \cap C[a,b].$$

Bohman's theorem is a particular case of Korovkin's theorem in which the functions $\theta_0 = e_0,\ \theta_1 = e_1,\ \theta_2 = e_2$ appear, and the operators are also of a particular form.

For the simultaneous approximation, the approximation of the functions together with their derivatives by linear and positive operators, an important property is the higher order convexity of the operators.

### 1.1.2 General evaluations with moduli of continuity $\omega_1$ and $\omega_2$

A simple way to estimate the degree of approximation with linear and positive operators is with the aid of the first order moduli of continuity. Such results were obtained in the works of O. Shisha and B. Mond [109] and of B. Mond [82].

**Theorem 1.1.4.** *[82] Let* $L : V \to \mathcal{F}(I)$ *be a linear and positive operator, where* $I$ *is a bounded interval and* $V$ *is a linear subspace of* $\mathcal{F}(I)$ *such that* $e_j \in V,\ j \in 0,1,2$*. For each* $g \in V,\ y \in I$ *and* $h > 0$ *we have*

$$\begin{aligned}|L(g,y) - g(y)| \leq & |g(y)||L(e_0,y) - 1| \\ & + \left(L(e_0,y) + \frac{1}{h^2} L((e_1 - ye_0)^2, y)\right) \cdot \omega_1(g,h). \qquad (1.2)\end{aligned}$$

The following estimates given by F. Altomare and M. Campiti [10] are based on the result established by O. Shisha and B. Mond in [109].

**Theorem 1.1.5.** *[109] Let* $L : C(I) \to \mathcal{F}(I)$ *be a linear and positive operator. It results the following:*

*i) If* $f \in C_B(I)$ *then* $(\forall)x \in I,\ (\forall)h > 0$

$$\begin{aligned}|L(f;x) - f(x)| \leq & |f(x)| \cdot |L(e_0;x) - 1| \\ & + \left(L(e_0;x) + h^{-1}\sqrt{L(e_0;x) \cdot L((e_1 - xe_0)^2;x)}\right)\omega_1(f;h).\end{aligned}$$

*ii) If* $f$ *is derivable on* $I$ *and* $f' \in C_B(I)$ *then* $(\forall)x \in I$ *and* $(\forall)h > 0$

$$\begin{aligned}&|L(f;x) - f(x)| \leq |f(x)| \cdot |L(e_0;x) - 1| + |f'(x)| \cdot |L(e_1;x) - xL(e_0;x)| \\ &+ \sqrt{L((e_1 - e_0x)^2;x)}\left(\sqrt{L(e_0;x)} + h^{-1}\sqrt{L((e_1 - xe_0)^2;x)}\right)\omega_1(f';h).\end{aligned}$$

In the category of 2nd order moduli we can include the module $(f,h) \to h\omega_1(f',h)$, $f \in \mathcal{D}(I)$. The coefficient $h$ in front of the 1st order moduli of the derivative is necessary due to the normalization condition.

An estimate given with this moduli can be seen in the following theorem:

**Theorem 1.1.6.** *[87] Let be $L : V \to \mathcal{F}(I)$, a linear and positive operator, where $V$ is a subspace of $\mathcal{F}(I)$. Suppose that $L(e_0; x) = e_0$ and $L(e_1; x) = e_1$. We have*

$$|L(f, x) - f(x)| \qquad \le \frac{1}{2}\sqrt{L\left((e_1 - xe_0)^2, x\right)} \cdot \omega_1\left(f', 2\sqrt{L\left((e_1 - xe_0)^2, x\right)}\right).$$

Estimates of the error involving the 2nd order smoothness moduli were obtained by H.H. Gonska [59] in the following theorem.

**Theorem 1.1.7.** *[59] Let be $a \le c < d \le b$ and $L : C[a, b] \to B[c, d]$ a positive and linear operator. For each $f \in C[a, b]$, $x \in [c, d]$ and $\delta > 0$ it holds the following inequalities:*

$$\begin{aligned}
&|L(f; x) - f(x)| \\
&\le \left[\frac{3}{2}\left(\|L\| + 1\right) + L\left((e_1 - xe_0)^2; x\right) \max\left\{\delta^{-2}; (b-a)^{-2}\right\}\right] \cdot \omega_2(f; \delta) \\
&+ 2|L(e_1 - xe_0; x)| \cdot \max\left\{\delta^{-1}, (b-a)^{-1}\right\} \cdot \omega_1(f; \delta) \\
&+ |L(e_0; x) - 1| \left(\|f\| + \omega_1(f; \delta)\right).
\end{aligned}$$

R. Păltănea [91] obtained an improved result with optimal constants, which expressed in terms of functional is the following:

**Theorem 1.1.8.** *Let $I$ be an arbitrary interval and let $F : C(I) \to \mathbb{R}$ be a linear and positive functional. Let $s \in \mathbb{N}$, $s \ge 2$. Then it holds*

$$\begin{aligned}
|F(f) - f(x)| \quad &\le \quad |f(x)| \cdot |F(e_0) - 1| + |F(e_1 - xe_0)| h^{-1} \omega_1(f, h) \\
&\quad + \Big(F(e_0) + \frac{1}{2} h^{-s} F(|e_1 - xe_0|^s)\Big) \omega_2(f, h), \qquad (1.3)
\end{aligned}$$

*for each $f \in C_B(I)$, $x \in I$ and $h > 0$ such that $h \preceq \frac{1}{2} lungime(I)$.*

Here the symbol $h \preceq \frac{1}{2} lungime(I)$ means $h \le \frac{1}{2} lungime(I)$ if $I$ is bounded interval and $h < \frac{1}{2} lungime(I)$ if $I$ is not a bounded interval.

### 1.1.3 Asymptotic results - Voronovskaya type theorems and saturation theorems

Consider the smallest concave majorant of $\omega(f, \cdot)$ given by the following formula

$$\tilde{\omega}_1(f, \epsilon) = \begin{cases} \sup\left\{\frac{(\epsilon - x)\omega(f, y) + (y - \epsilon)\omega(f, x)}{y - x} : 0 \le x \le \epsilon \le y \le b - a \ x \ne y\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 \le \epsilon \le b - a \\ \tilde{\omega}(f, b - a) = \omega(f, b - a) \quad \text{if } \epsilon > b - a \end{cases}$$

A general quantitative result of type Voronokaja is the following:

**Theorem 1.1.9.** *[55] Let $L : C[0, 1] \to C[0, 1]$ a linear and positive operator such that $L(e_j; x) = e_j$ for $j = 0, 1$. If $f \in C^2[0, 1]$ and $x \in [0, 1]$ then*

$$\begin{aligned}
\left|L(f; x) - f(x) - \frac{1}{2} \cdot f''(x) L\left((e_1 - x)^2; x\right)\right| \le &\frac{1}{2} L\left((e_1 - x)^2; x\right) \\
&\cdot \tilde{\omega}\left(f''; \frac{1}{3} \cdot \sqrt{\frac{L\left((e_1 - x)^4; x\right)}{L\left((e_1 - x)^2; x\right)}}\right).
\end{aligned}$$

A general asymptotic theorem is the following:

**Theorem 1.1.10.** *[78] Let be $q \in \mathbb{N}$ fixed, $f \in C^q[0,1]$ and $L_n : C[0,1] \to C[0,1]$ a sequence of linear and positive operators so that*

$$L_n(e_0;x) = 1, \qquad x \in [0,1]$$
$$\lim_{n\to\infty} \frac{L_n\left((e_1-x)^{q+2j};x\right)}{L_n\left((e_1-x)^q;x\right)} = 0, \qquad \textit{for at least one} \quad j \in \{1,2,\cdots\}$$

*Then*

$$\frac{1}{L_n\left((e_1-x)^q;x\right)}\left\{L_(f;x) - f(x) - \sum_{r=1}^{q} L_n\left((e_1-x)^r;x\right)\cdot\frac{f^{(n)}(x)}{r!}\right\} \to 0$$

*cand $n \to \infty$.*

A quantitative form of the previous theorem is the following:

**Corollary 1.1.1.** *[55] Let be $q \in \mathbb{N}_0$, $f \in C^q[0,1]$ and $L : C[0,1] \to C[0,1]$ a linear and positive operator. Then*

$$\left|L(f;x) - \sum_{r=0}^{q} L\left((e_1-x)^r;x\right)\frac{f^{(r)}(x)}{r!}\right| \leq \frac{L\left(|e_1-x|^q;x\right)}{q!}$$
$$\cdot\,\tilde{\omega}_1\left(f^{(q)};\frac{1}{q+1}\cdot\frac{L\left(|e_1-x|^{q+1};x\right)}{L\left(|e_1-x|^q;x\right)}\right).$$

At the end of this section, we mention a result of asymptotic behavior that uses a weighted moduli.

We will consider in the following the weight $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. Let be $\Psi(x) = x(1-x)$, $x \in [0,1]$. Let me $\mu \in [0,1]$. We define

$$\omega_2^{\varphi,\mu}(f,h)$$
$$= \sup\left\{\Psi^{1-\mu}(x)|f(x-\varphi) - 2f(x) + f(x+\varphi)|,\ x \pm \rho \in [0,1],\ |\rho| \leq h\rho^{\mu}(x)\right\}$$

where $f \in B[0,1]$, $h > 0$.

**Theorem 1.1.11.** *[88] Let be $\Psi(x) = x(1-x)$, $x \in [0,1]$ and let be $(L_n)_n$ a sequence of linear and positive operators $L_n : B[0,1] \to \mathbb{R}^{[0,1]}$ which satisfies the following conditions*

*i)* $L_n(e_i) = e_i$, $i = 0,1$

*ii)* *it exists a sequence $(\alpha_n)_n$ such that $\alpha_n > 0$, $\lim_{n\to\infty}\alpha_n = 0$ and*

$$\lim_{n\to\infty}\frac{L_n\left((e_1-xe_0)^2,x\right)}{\alpha_n\Psi(x)} = 1 \ \textit{uniform in raport cu}\ x \in (0,1)$$

*iii)* $L_n\left((e_1-xe_0)^4,x\right) = o\left(L_n\left((e_1-xe_0)^2,x\right)\right)$ $(n \to \infty)$ *uniformly related to* $x \in [0,1]$.

*Then for ecah $\mu \in [0,1]$ and each $f \in C^2[0,1]$*

$$\lim_{n\to\infty}\frac{|]|L_nf - f||}{\omega_2^{\varphi,\mu}(f,\sqrt{\alpha_n}} = \frac{1}{2}.$$

## 1.2 Asymptotic evaluations with optimal constants of linear and positive operators in relation to 1st and 2nd order moduli of continuity

In this section we will aim to give a general estimate of asymptotic developments using the moduli of continuity and to point out the optimality of the constants that appear in these estimates. Results from this section are included in the paper [95].

For simplicity, instead of a sequence of linear and positive operators it is sufficient to consider a single such operator $L$. We will present two types of estimates. The first estimate is for the amount of shape:

$$L\Big(f-\sum_{j=0}^{q}\frac{f^{(j)}(x)}{j!}(e_1-x)^j,x\Big)=L(f,x)-\sum_{j=0}^{q}\frac{f^{(j)}(x)}{j!}L((e_1-x)^j,x),$$

where $q$ is even.

The second is for a quantity of form:

$$L\Big(\frac{k!}{(e_1-x)^k}\cdot\Big(f-\sum_{j=0}^{k}\frac{f^{(j)}(x)}{j!}(e_1-x)^j\Big),x\Big),$$

for any natural $k$, where the function to which we apply the operator is extended by continuity in the point $x$.

In the next sub-chapter we will exemplify this theory for Bernstein type operators, for which we have the opportunity to compare the results with those obtained previously.

### 1.2.1 A first type of asymptotic estimation using the moduli of continuity

Let $I\subset\mathbb{R}$ be an arbitrary interval.

Let $L:C(I)\to\mathcal{F}(I)$ a linear and positive operator. For $x\in I$ and $j\in\mathbb{N}_0$ we consider

$$m_j(x)=L((e_1-xe_0)^j,x),$$
$$M_j(x)=L(|e_1-xe_0|^j,x).$$

Then $M_{2k}(x)=m_{2k}(x)$, for $k\in\mathbb{N}$.

**Theorem 1.2.1.** *Let $L:C(I)\to\mathcal{F}(I)$ a linear and positive operator, where $I$ is an interval. For $k\in\mathbb{N}_0$, $s\in\mathbb{N}$, $f\in C_B^k(I)$, $x\in I$ and $h>0$ one has*

$$\Big|L(f,x)-\sum_{j=0}^{k}\frac{f^{(j)}(x)}{j!}m_j(x)\Big|\le\Big(\frac{1}{k!}M_k(x)+h^{-s}\frac{s!}{(k+s)!}M_{k+s}(x)\Big)\omega_1(f^{(k)},h).\quad(1.4)$$

*For $s=1$ and every $k\in\mathbb{N}_0$ the constants given in (1.4) are optimal, i.e. if there are two constants $A>0$, $B>0$, so that there holds the following inequality:*

$$\Big|L(f)(x)-\sum_{j=0}^{k}\frac{f^{(j)}(x)}{j!}m_j(x)\Big|\le\Big(AM_k(x)+Bh^{-1}M_{k+1}(x)\Big)\omega_1(f^{(k)},h)\quad(1.5)$$

*for any linear and positive operator $L:C(I)\to\mathcal{F}(I)$, $(\forall)f\in C_B^k(I)$, $(\forall)x\in I$ and $(\forall)h>0$, then $A\ge\frac{1}{k!}$ and $B\ge\frac{1}{(k+1)!}$.*

**Corollary 1.2.1.** *Let $L : C(I) \to \mathcal{F}(I)$ be a linear and positive operator. Let be $x \in I$ and $k \in \mathbb{N}_0$, $s \in \mathbb{N}$. Suppose that $M_j(x) \neq 0$, for $j = k, k+s$. Then for any function $f \in C_B^k(I)$ it holds*

$$\left| L(f,x) - \sum_{j=0}^{k} \frac{f^{(j)}(x)}{j!} m_j(x) \right| \leq 2 \frac{M_k(x)}{k!} \omega_1 \left( f^{(k)}, \sqrt[s]{\frac{s!k!M_{k+s}(x)}{(k+s)!M_k(x)}} \right). \tag{1.6}$$

Corollary 1.2.1 was obtained for Bernstein operators for $s = 1$ in [2].

### 1.2.2 The first type of asymptotic estimates using first and second order moduli of continuity

We will use the following auxiliary lemma, which is a variant of the result given in [87], Lemma 1.1.1.

**Lemma 1.2.1.** *Let be a linear and positive function $F : C(I) \to \mathbb{R}$.*
*i) We assume that it exists $x \in I$ and $\sigma \in C(I)$, with the following properties*

*a) $\sigma(x) = 0$ and $\sigma(t) > 0$, for each $t \in I \setminus \{x\}$;*

*b) $F(\sigma) = 0$.*

*Then $F(f) = F(e_0)f(x)$, $f \in C(I)$.*
*ii) If $F(e_0) = 0$, then $F = 0$.*

The main result of this sub-chapter is the following theorem:

**Theorem 1.2.2.** *Let $I$ be a real interval. Let $L : C(I) \to \mathcal{F}(I)$ be a positive linear operator. Let be $s \in \mathbb{N}$, $s \geq 2$. Let be $k \in \mathbb{N}_0$. For each $f \in C_B^{2k}(I)$, $x \in I$ and $h > 0$ such that $lungime(I) \geq 2h$ it holds:*

$$\begin{aligned}\left| L(f,x) - \sum_{j=0}^{2k} \frac{f^{(j)}(x)}{j!} m_j(x) \right| \leq & \frac{|m_{2k+1}(x)|}{(2k+1)!} h^{-1} \omega_1(f^{2k)}, h) \\ & + \Big( \frac{m_{2k}(x)}{(2k)!} + h^{-s} \frac{s!}{2} \cdot \frac{M_{2k+s}(x)}{(2k+s)!} \Big) \omega_2(f^{(2k)}, h). \end{aligned} \tag{1.7}$$

*For $s = 2$ and any $k \in \mathbb{N}_0$ the constants given by the relation (1.7) are optimal, i.e. if there are three constants $A > 0$, $B > 0$, $C > 0$ such that the following inequality:*

$$\begin{aligned}\left| L(f,x) - \sum_{j=0}^{2k} \frac{f^{(j)}(x)}{j!} m_j(x) \right| \leq & A|m_{2k+1}(x)| h^{-1} \omega_1(f^{2k)}, h) \\ & + \Big( Bm_{2k}(x) + Ch^{-2} m_{2k+2}(x) \Big) \omega_2(f^{(2k)}, h) \end{aligned} \tag{1.8}$$

*it is true for any linear and positive operator $L : C(I) \to \mathcal{F}(I)$, $(\forall) f \in C_B^{2k}(I)$, $(\forall) x \in I$ and $(\forall) h > 0$ such that $lungime(I) \geq 2h$, then*

$$A \geq \frac{1}{(2k+1)!},\ B \geq \frac{1}{(2k)!},\ C \geq \frac{1}{(2k+2)!}. \tag{1.9}$$

### 1.2.3 A second type of asymptotic estimates

If $f \in C^k(I)$, $k \in \mathbb{N}_0$, $x \in I$, then the function

$$\Delta_x^k(f)(t) = \begin{cases} \frac{k!}{(t-x)^k}\left(f(t) - \sum_{j=0}^{k} \frac{f^{(j)}(x)}{j!}(t-x)^j\right), & t \in I,\ t \neq x \\ 0, & t = x \end{cases} \tag{1.10}$$

is continuous on $I$.

The objective of this section is to give estimates for $L(\Delta_x^k(f), x)$, when $L : C(I) \to C(I)$ is a linear and positive operator.

For $L$ and $k \in \mathbb{N}$ we define the following operator $V_{L,k} : C_B(I) \to \mathcal{F}(I)$:

$$V_{L,k}(g, x) := L\left(\theta_x^k(g), x\right)\left(g \in C_B(I),\ x \in I\right), \tag{1.11}$$

where function $\theta_x^k(g) \in C_B(I)$ is given by the relation

$$\theta_x^k(g)(t) = \begin{cases} \frac{k}{(t-x)^k}\int_x^t (t-u)^{k-1} g(u)\mathrm{d}u, & t \in I,\ t \neq x \\ g(x), & t = x \end{cases} \tag{1.12}$$

**Lemma 1.2.2.** *For any linear and positive operator $L : C(I) \to C(I)$ and any $k \in \mathbb{N}$, the operator $V_{L,k}$ is well defined, linear and positive.*

**Lemma 1.2.3.** *If $L : C(I) \to C(I)$ is a linear and positive operator, then*

$$L\left(\Delta_x^k(f), x\right) = V_{L,k}(f^{(k)}, x) - L(e_0, x) f^{(k)}(x), \tag{1.13}$$

*for $f \in C^k(I)$, $k \in \mathbb{N}$, $x \in I$.*

**Lemma 1.2.4.** *For any linear and positive operator $L : C(I) \to C(I)$, $k \in \mathbb{N}$ and each $s \in \mathbb{N}_0$ one has*

$$V_{L,k}(|e_1 - x|^s, x) = \frac{s!k!}{(k+s)!} L(|e_1 - x|^s, x). \tag{1.14}$$

**Theorem 1.2.3.** *Let $L : C(I) \to C(I)$ be a linear and positive operator, $s \in \mathbb{N}$, $k \in \mathbb{N}_0$. For $f \in C^k(I)$, $x \in I$ and $h > 0$ one has:*

$$\left|L\left(\Delta_x^k(f), x\right)\right| \quad \leq \quad \left(L(e_0, x) + \frac{s!k!}{(k+s)!} h^{-s} L(|e_1 - x|^s)\right)\omega_1(f^{(k)}, h). \tag{1.15}$$

*For $s = 1$ and any $k \in \mathbb{N}_0$ the constants given by the relation (1.15) are optimal, i.e. if there are two constants $A > 0$, $B > 0$, so that inequality:*

$$|L(\Delta_x^k(f), x)| \leq \left(AL(e_0, x) + Bh^{-1} L(|e_1 - x|, x)\right)\omega_1(f^{(k)}, h). \tag{1.16}$$

*it is true for any linear and positive operator $L : C(I) \to C(I)$, $(\forall) f \in C^k(I)$, $(\forall) x \in I$ and $(\forall) h > 0$ such that $lungime(I) \geq 2h$, then*

$$A \geq 1,\ B \geq \frac{1}{k+1}. \tag{1.17}$$

**Theorem 1.2.4.** *For $L : C(I) \to C(I)$, a linear and positive operator, $s \in \mathbb{N}$, $s \geq 2$, $k \in \mathbb{N}_0$, $f \in C^k(I)$, $x \in I$ and $h > 0$ such that $lungime(I) \geq 2h$ it holds:*

$$\begin{aligned} \left|L(\Delta_x^k(f), x)\right| &\leq \frac{1}{k+1} |L(e_1 - x, x)| h^{-1} \omega_1(f^{(k)}, h) \\ &\quad + \left(L(e_0, x) + \frac{s!k!}{2(k+s)!} h^{-s} L(|e_1 - x|^s, x)\right)\omega_2(f^{(k)}, h). \end{aligned} \tag{1.18}$$

*For $s=2$ and any $k\in\mathbb{N}_0$ the constants given by relation (1.18) are optimal, i.e. if there are three constants $A>0$, $B>0$, $C>0$ so that the following inequality:*

$$\begin{aligned}\left|L(\Delta_x^k(f),x)\right| &\leq A|L(e_1-x,x)|h^{-1}\omega_1(f^{(k)},h)\\ &+\Big(BL(e_0,x)+Ch^{-2}L((e_1-x)^2,x)\Big)\omega_2(f^{(k)},h). \qquad (1.19)\end{aligned}$$

*it is true for any linear and positive operator $L:C(I)\to C(I)$, $(\forall)f\in C^k(I)$, $(\forall)x\in I$ and $(\forall)h>0$ such that $lungime(I)\geq 2h$, then*

$$A\geq\frac{1}{k+1},\ B\geq 1,\ C\geq\frac{1}{(k+1)(k+2)}. \qquad (1.20)$$

# Chapter 2

# Bernstein type operators

## 2.1 Bernstein operators

We will give the following definition of Bernstein operators which we will continue to use.

**Definition 2.1.1.** *Let be $n \in \mathbb{N}$. The operators $B_n : C[0,1] \to C[0,1],\ f \mapsto B_n f$ defined by*

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right) \tag{2.1}$$

*where*

$$p_{n,k}(x)\binom{n}{k} x^k (1-x)^{n-k},\ k = \overline{0,n},\ x \in [0,1] \tag{2.2}$$

*they are named Bernstein operators. $B_n f$ is the Bernstein polynomial of degree $n$ and $(p_{n,k})_{k=\overline{0,n}}$ represents the fundamental polynomials of Bernstein of degree $n$.*

With the aid of Bernstein's polynomials, a constructive procedure was obtained to prove Weierstrass's approximation theorem. 1.1.1).

**Theorem 2.1.1.** *[10] For a function $f(x)$ bounded on $[0,1]$ it holds*

$$\lim_{n\to\infty} B_n(f,x) = f(x)$$

*for any point of continuity $x$ of $f$; and the relation holds uniformly on $[0,1]$ if $f(x)$ este continues on this interval.*

**Theorem 2.1.2.** *[10] The operators $(B_n)_{n\in\mathbb{N}}$ represented in the Definition 2.1.1 have the following properties:*

1) *The operators $B_n$ are linear and positive.*
2) $\sum_{k=0}^{n} p_{n,k}(x) = 1$.
3) $B_n(e_0;x) = 1,\ B_n(e_1;x) = x,\ B_n(e_2;x) = x^2 + \frac{x(1-x)}{n}$.
4) $\|B_n\| = 1$.
5) *Bernstein's polynomial is relative to $f$ and the nodes $0$ and $1$, meaning that $B_n(f;0) = f(0),\ B_n(f;1) = f(1)$.*

*7) A stronger result was given in 1954 by W. B. Temple [121] and O. Arama [16] who tells us that for any convex function $f$ on $[0,1]$ we have inequality*

$$B_{n+1}(f;x) \leq B_n(f;x),\ x \in [0,1],\ n \in \mathbb{N}.$$

*6) Bernstein operators preserve linear functions and the equality holds*

$$f \leq B_n(f),\ n \in \mathbb{N}$$

*for any function $f$ convex.*

The $r \geq 1$ order derivatives of Bernstein polynomials can be expressed as follows:

$$B_n^{(r)}(f;x) = \frac{r!n!}{(n-r)!n^r} \sum_{k=0}^{n-r} \left[f; \frac{k}{n}, \frac{k+1}{n}, \cdots, \frac{k+r}{n}\right] p_{n-r,k}(x) \tag{2.3}$$

This result was given by Popoviciu [101], which shows that for $f$ convex of the order $k \geq 1$, $B_n(f)$ is also convex. S. Wigert [129] first showed that

$$\lim_{n\to\infty} ||B_n^{(r)}(f) - f^{(r)}|| = 0 \text{ daca } f \in C^r[0,1],\ r \geq 1.$$

### 2.1.1 Evaluations with moduli of continuity. Optimality of constants

The first estimate of Bernstein's approximation was given by T. Popoviciu [99] using the first moduli of continuity

$$||B_n(f) - f|| \leq \frac{3}{2}\omega_1\left(f, \frac{1}{\sqrt{n}}\right),\ f \in C[0,1],\ n \in \mathbb{N}.$$

The optimal estimate is obtained by P. C. Sikkema[113]

$$\sup_{f\in C[0,1]\setminus\Pi_0} \sup_{n\in\mathbb{N}} \frac{||B_n(f) - f||}{\omega_1\left(f, \frac{1}{\sqrt{n}}\right)} = \frac{4306 + 837\sqrt{6}}{5832} = 1,08988\cdots$$

We also have the following asymptotic constant, given by G. C. Essen [48]

$$\sup_{f\in C[0,1]\setminus\Pi_0} \limsup_{n\to\infty} \frac{||B_n(f) - f||}{\omega_1\left(f, \frac{1}{\sqrt{n}}\right)} = 2\sum_{k=0}^{\infty}(k+1)\left(\lambda(2k+2) - \lambda(2k)\right) = 1,04556\cdots$$

where $\lambda(x) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{x} \exp^{-\frac{t^2}{2}} dt$.

According to B. Mond [82], we have

$$|B_n(f;x) - f(x)| \leq 2\omega_1\left(f, \sqrt{\frac{x(1-x)}{n}}\right),\ f \in B[0,1],\ n \in \mathbb{N},\ x \in (0,1).$$

In the case of differential functions with continuous derivative, the first estimate with the first order moduli was given by T. Popoviciu [99]. The punctual version can be given in the form:

$$|B_n(f;x) - f(x)| \leq C\sqrt{\frac{x(1-x)}{n}}\omega_1\left(f', 2\cdot\sqrt{\frac{x(1-x)}{n}}\right),\ f \in C^1[0,1]$$

where $x \in (0,1)$, $n \in \mathbb{N}$ and $C > 0$ it is an absolute constant. The best value of the constant $C$ in this inequality is $C = \frac{1}{2}$ and was obtained in [87]. From this estimate can be obtained the best global approximation first found by F. Schurr and W. Steute [112].

$$\sup_{f \in C[0,1] \setminus \Pi_1} \sup_{n \in \mathbb{N}} \frac{||B_n(f) - f||}{\frac{1}{\sqrt{n}} \omega_1 \left( f', \frac{1}{\sqrt{n}} \right)} = \frac{1}{4}$$

Outstanding estimates with moduli of continuity of order 2 are given by Y.A. Brudnyi[23] which proved that there is a constant $C > 0$ such that

$$||B_n f - f|| \leq C\omega_2 \left( f, \frac{1}{\sqrt{n}} \right), \ f \in C[0,1], \ n \in \mathbb{N}. \tag{2.4}$$

The punctual version of this result is

$$|B_n(f;x) - f(x)| \leq C\omega_2 \left( f, \sqrt{\frac{x(1-x)}{n}} \right), \ f \in C[0,1], \ n \in \mathbb{N}, \ x \in (0,1). \tag{2.5}$$

which was first obtained by Jia - ding Cao [69].

The first concrete constants to verify (2.4) are given by H. Gonska in [54] and [57], where we have the constant $C = 3.25$ for (**??**) and the constant $C = 1.6875$ for (2.4) obtained together with R. K. Kovacheva [58].

In [92] R. Păltănea proved that $C = 1$ is the best possible constant for (2.4), and in [93] it was shown that $C = \frac{3}{2}$ is the best constant that can appear in (2.5).

### 2.1.2 Asymptotic evaluations for Bernstein's operators

For Bernstein operators, Voronoskaya proved in [127] the theorem that has his name:

**Theorem 2.1.3.** *[127] If $f$ is bounded on $[0,1]$, differentiable on a closeness of his $x$ and admits the derivative of order 2, $f''(x)$ for $x \in [0,1]$ then*

$$\lim_{n \to \infty} n \left[ B_n(f;x) - f(x) \right] = \frac{x(1-x)}{2} f''(x).$$

*If $f \in C^2[0,1]$, then the convergence is uniform.*

This result has been in the attention of many mathematicians over the past few decades. Voronoskaya's result was the inspiration of S. Bernstein to generalize it as follows:

**Theorem 2.1.4.** *[20] If $q \in \mathbb{N}$ is even, $f \in C^q[0,1]$, the uniformly for $x \in [0,1]$ we have*

$$n^{\frac{q}{2}} \left[ B_n(f;x) - f(x) - \sum_{r=1}^{q} B_n\left( (e_1 - xe_0)^r; x \right) \cdot \frac{f^{(r)}(x)}{r!} \right] \to 0$$

The theorem 2.1.4 has recently been generalized for any $q$ index by Tachev.

**Theorem 2.1.5.** *[120] Let be $q \in \mathbb{N}$ and $C^q[0,1]$. Then there is no constant $\beta > 0$ so that the uniform convergence of the relation holds*

$$n^{\frac{q}{2}+\beta} \left[ B_n(f;x) - f(x) - \sum_{r=1}^{q} B_n\left( (e_1 - xe_0)^r; x \right) \cdot \frac{f^{(r)}(x)}{r!} \right] \to 0, \ n \to \infty$$

We present below estimates that can be obtained as applications to the theory presented in Chapter 1.2 at Bernstein operators. These results were obtained in our paper [95]. For this we need the following result:

$$B_n((e_1-x)^{2s},x) \le 4x(1-x)\frac{(2s)!}{8^s s! n^s},\ x\in[0,1],\ s\in\mathbb{N},\ n\in\{1,2,3,4,5\}. \tag{2.6}$$

This result was obtained by D. Cardenas - Morales [33] for $n\in\{1,2,3,4\}$ and by J. Adell and D. Cardenas - Morales în [4] for $n=5$. From this result the Theorem 1.2.1 with $2s$, instead of $s$ we obtain

**Theorem 2.1.6.** *[95] For $f\in C^2[0,1]$, $k\in\mathbb{N}_0$, $n,s\in\mathbb{N}$, $x\in(0,1)$ and $h>0$ it holds*

$$\begin{aligned}&\Big|B_n(f)(x)-\sum_{j=0}^{2k}\frac{f^{(j)}(x)}{j!}B_n((e_1-x)^j)(x)\Big|\\ &\qquad\le \frac{4x(1-x)}{8^k n^k}\Big(\frac{1}{k!}+h^{-2s}\frac{(2s)!}{8^s(s+k)!n^s}\Big)\omega_1(f^{(2k)},h).\end{aligned} \tag{2.7}$$

V. S. Videnski ĭ [125] showed that for $C=1$ the following inequality holds

$$\Big|B_n(f,x)-f(x)-\frac{x(1-x)}{2n}\cdot f''(x)\Big|\le C\frac{x(1-x)}{n}\omega_1\Big(f'',\sqrt{\frac{2}{n}}\Big), \tag{2.8}$$

which holds for any $f\in C^2[0,1]$, $x\in(0,1)$, $n\in\mathbb{N}$, $n\ge 2$. This inequality was called Videnskyi-type inequality for Bernstein's operators. There are many improvements about the constant $C$ that can occur in this inequality: Gonska and Raşa [63] got $C=1/2+\sqrt{5/16}=0.895285\ldots$, Abel and Siebert citeAbelSiebert1 obtained $C=11/16=0.6875$, C árdenas-Morales [33] gave $C=617/1024=0.6025390625$ and recently Adell and C árdenas-Morales [3] obtained $C=1/2+15/(16)^3=0.503662109375\ ldots$.

Using Theorem 2.1.6 we can deduce that we have:

**Corollary 2.1.1.** *[95] For each $f\in C^2[0,1]$, $n\ge 2$, $x\in[0,1]$ one has*

$$\Big|B_n(f,x)-f(x)-\frac{x(1-x)}{2n}\cdot f''(x)\Big|\le C_0\frac{x(1-x)}{n}\omega_1\Big(f'',\sqrt{\frac{2}{n}}\Big), \tag{2.9}$$

*where*

$$C_0=\frac{1}{2}\left(1+\frac{10!}{16^5 6!}\right)=0,50240325927734375.$$

**Remark 2.1.1.** *We choose $s=5$ because for this value the minimum expression is obtained $\frac{(2s)!}{16^s(s+1)!}$. This is the best value obtained for Vidensky's constant.*

On the other hand, it does not result that $C_0$ is the best possible constant, because the estimate given in (2.7) is not optimal.

**Corollary 2.1.2.** *[95] For any $f\in C^{2k}[0,1]$, $n\ge 2$, $k\ge 1$, $x\in[0,1]$ one has*

$$\begin{aligned}&\Big|B_n(f,x)-\sum_{j=0}^{2k}\frac{f^{(j)}(x)}{j!}B_n((e_1-x)^j,x)\Big|\\ &\le\frac{4x(1-x)}{8^k k! n^k}\left[\frac{1}{2\sqrt{2k+1}}\omega_1\left(f^{(2k)},\frac{1}{\sqrt{n}}\right)+\Big(1+\frac{1}{8(k+1)}\Big)\omega_2\left(f^{(2k)},\frac{1}{\sqrt{n}}\right)\right].\end{aligned} \tag{2.10}$$

No other asymptotic evaluations with the $\omega_2$ module are known in the literature with which to compare this result.

## 2.2 Other modified Bernstein operators

In this paragraph we consider some of the most well-known operators obtained by modifying Bernstein's operators.

### 2.2.1 Kantorovich operator

**Definition 2.2.1.** *Let be $n \in \mathbb{N}$. The operators $K_n : L_1[0,1] \to C[0,1],\ f \mapsto K_n f$ defined by*

$$(K_n(f;x) = (n+1)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt$$

*is named Kantorovich operator.*

If we note by $\chi_n$ the characteristic function of the interval $\left(0, \frac{1}{n+1}\right]$ then the Kantorovich operators can be expressed as follows:

$$(K_n(f;x) = (n+1)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int_0^1 f(t)\chi_n\left(t-\frac{k}{n+1}\right)dt.$$

**Lemma 2.2.1.** *[5] Kantorovich operators given by the Definition 2.2.1 verify the relations:*

*i)* $K_n(e_0;x) = 1.$

*ii)* $K_n(e_1;x) = \frac{n}{n+1}x + \frac{1}{2(n+1)}.$

*iii)* $K_n(e_2;x) = \frac{n(n-1)}{(n+1)^2}x^2 + \frac{2n}{(n+1)^2}x + \frac{1}{3(n+1)^2}.$

*iv)* $\lim_{n\to\infty} K_n f = f$ *uniformly on* $[0,1]$, *for any* $f \in C[0,1]$.

From Shisha and Mond's estimate we obtain:

**Theorem 2.2.1.** *If $f \in C[0,1]$ and $x \in [0,1]$ then*

$$\begin{aligned}|K_n(f;x) - f(x)| &\le 2\omega_1\left(f, \frac{\sqrt{(n-1)x(1-x)+\frac{1}{3}}}{n+1}\right)\\ &\le 2\omega_1\left(f; \frac{1}{2\sqrt{n+1}}\right).\end{aligned}$$

### 2.2.2 Durrmeyer operator

**Definition 2.2.2.** *Let be $n \in \mathbb{N}$. The operators $D_n : L_1[0,1] \to C[0,1],\ f \mapsto D_n f$ defined by*

$$D_n(f;x) = (n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt$$

*where $p_{n,k}$ are Bernstein's fundamental polynomials, called the Durrmeyer operator.*

It is observed that they are linear and positive. The following properties were obtained by M.M. Dierrennic [43], see also O. Agratini [5].

**Theorem 2.2.2.** *Durrmeyer operators have the following properties:*

1) $D_n(e_0) = 1$.

2) *Transforms any polynomial of degree p, $p \leq n$ in a polynomial of degree p.*

3) $\lim_{n\to\infty} D_n f = f$ *uniformly on* $[0,1]$, *for any* $f \in C[0,1]$.

4) $|D_n(f;x) - f(x)| \leq 2\omega_1\left(f, \frac{1}{\sqrt{2n+6}}\right), \ (\forall) f \in C[0,1], (\forall) n \geq 3$.

### 2.2.3 Durrmeyer–genuine operator

We consider the operator Bernstein - Durrmeyer - genuine $U_n : C[0,1] \to \prod_n$, considered by W. Chen [35] and T. N. T. Goodman si A. Sharma [64].

$$U_n(f;x) = f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k-1}(t)f(t)dt.$$

By analogy with well-known Bernstein - Durrmeyer operators, Bernstein - Durrmeyer - genuine operators have the following properties:

**Lemma 2.2.2.** *[64][65] The following properties hold:*

(i) $U_n$ *is linear and positive.*

(ii) $U_n e_i = e_i, \ i = \overline{0,1}$ *and* $U_n(e_2;x) = x^2 + \frac{2x(1-x)}{n+1}$.

(iii) $\|U_n f\| \leq \|f\|$.

(iv) $f \leq U_k f \leq U_n f$ *for* $f$ *convex and natural number* $k \geq n$.

(v) $U_n$ *is of diminished variation, that is, for any* $f \in C[0,1]$ *a nonlinear function* $S(U_n f - f) \leq S(f - f)$, *where* $S(g)$ *is the function that counts the number of sign changes of the function* $g$ *on* $[0,1]$.

We consider the differential operators

$$A := \frac{x(1-x)}{2} \cdot \frac{d^2}{dx^2}, \quad B := 2A = x(1-x)\frac{d^2}{dx^2} \tag{2.11}$$

with the common domain

$$D(A) = D(B) = \left\{u \in C[0,1] \cap C^2[0,1] : \lim_{x\to 0} x(1-x)u''(x) = \lim_{x\to 1} x(1-x)u''(x) = 0\right\}$$

**Theorem 2.2.3.** *[60] For any function* $f \in C^2[0,1]$ *it holds:*

$$\lim_{n\to\infty} n(B_n f - f) = Af.$$

**Theorem 2.2.4.** *It exists the value*

$$\|U_n f - f\| \leq \frac{9}{8}\omega_2\left(f, \frac{1}{\sqrt{n+1}}\right), \quad f \in C[0,1], \ n \in \mathbb{N}. \tag{2.12}$$

### 2.2.4 $\alpha$–Bernstein operators

In [34], the authors introduced a new family of modified Bernstein operators.

**Definition 2.2.3.** *[34] We consider the function $f$ defined on $[0,1]$, for any natural number $n$ and any $\alpha \in \mathbb{R}$ fixed, we define the operator $\alpha$-Bernstein for $f$*

$$T_{n,\alpha}(f;x) = \sum_{i=0}^{n} f_i p_{n,i}^{(\alpha)}(x) \tag{2.13}$$

*where $f_i = f\left(\frac{i}{n}\right)$. For $i = 0,1,\ldots,n$, the $\alpha$-Bernstein polynomial $p_{n,i}^{\alpha}(x)$ of degree $n$ is define as $p_{1,0}^{\alpha}(x) = 1-x$, $p_{1,1}^{\alpha}(x) = x$ and*

$$\begin{aligned} p_{n,i}^{(\alpha)}(x) = & \binom{n-2}{i}(1-\alpha)x + \binom{n-2}{i-2}(1-\alpha)(1-x) \\ & + \binom{n}{i}\alpha x(1-x)\cdot x^{i-1}(1-x)^{n-i-1}, \end{aligned} \tag{2.14}$$

*where $n \geq 2$, $0 \leq i \leq n$, $x \in [0,1]$.*

Calculating some terms as follows, we observe the following formulas:

$$\begin{aligned} p_{n,0}^{\alpha}(x) &= (1-\alpha x)(1-x)^{n-1} \\ p_{n,n}^{\alpha}(x) &= (1-\alpha+\alpha x)x^{n-1} \end{aligned} \tag{2.15}$$

For $\alpha = 1$, the $\alpha$-Bernstein polynomial reduces to the classical Bernstein polynomial.

**Lemma 2.2.3.** *[34] For any $n \geq 1$, and any $\alpha$:*

*i) (Interpolation property) The $\alpha$-Bernstein operator for $f(x)$ interpolates $f(x)$ at both ends of the range $[0,1]$, i.e.*

$$T_{n,\alpha}(f;0) = f(0) \ si \ T_{n,\alpha}(f;1) = f(1).$$

*ii) (Linearity) The $\alpha$-Bernstein operator is linear, otherwise*

$$T_{n,\alpha}(\lambda f + \mu g) = \lambda T_{n,\alpha}(f) + \mu T_{n,\alpha}(g),$$

*for any function $f(x)$ and $g(x)$ define on $[0,1]$ and any $\lambda, \mu \in \mathbb{R}$.*

**Lemma 2.2.4.** *[34] The following properties hold:*

*i) $T_{n,\alpha}(e_0;x) = 1$.*

*ii) $T_{n,\alpha}(e_1;x) = x$.*

*iii) The $\alpha$-Bernstein operator reproduces the linear functions as follows*

$$T_{n,\alpha}(ax+b;x) = ax+b$$

*for any real numbers $a$ and $b$.*

*iv) The $\alpha$-Bernstein operator is a monotone operator for $0 \leq \alpha \leq 1$, so, if $f(x) \geq g(x)$, $x \in [0,1]$, then $T_{n,\alpha}(f,x) \geq T_{n,\alpha}(g;x)$, $x \in [0,1]$.*

*v) If $f(x)$ is negative on $[0,1]$, then $\alpha$-Bernstein operator it also turns it into a positive function.*

**Lemma 2.2.5.** *[34] The following identities hold:*

*i)* $T_{n,\alpha}(e_2;x) = x^2 + \frac{n+2(1-\alpha)}{n^2}x(1-x)$.

*ii)* $T_{n,\alpha}(e_3;x) = x^3 + \frac{3[n+2(1-2\alpha)]}{n^2}x^2(1-x) + \frac{n+6(1-\alpha)}{n^3}x(1-x)(1-2x)$.

*iii)*

$$\begin{aligned} T_{n\alpha}(e_4;x) =& x^4 + \frac{6[n+2(1-\alpha)]}{n^2}x^3(1-x) + \frac{4[n+6(1-\alpha)]}{n^3}x^2(1-x)(1-2x) \\ &+ \frac{[3n(n-2_+12(n-6)(1-\alpha]x(1-x)+[n+14(1-\alpha)]}{n^4}x(1-x). \end{aligned}$$

**Theorem 2.2.5.** *[34] If function $f(x)$ is continuous on $[0,1]$, for any $\alpha \in [0,1]$, the $\alpha$-Bernstein operator converges uniformly to $f(x)$ over the interval $[0,1]$.*

The following theorem is a Voronoskaya type theorem.

**Theorem 2.2.6.** *[34] Let $f(x)$ be bounded on $[0,1]$. Then, for each $x \in [0,1]$ where $f''(x)$ exists,*

$$\lim_{n\to\infty} n[T_{n,\alpha}(f;x) - f(x)] = \frac{1}{2}x(1-x)f''(x),$$

*where $0 \le \alpha \le 1$.*

The following result gives us a higher margin for the approximation error.

**Theorem 2.2.7.** *[34] If $f(x)$ is bounded on $[0,1]$, then, for $0 \le \alpha \le 1$*

$$||f(x) - T_{n,\alpha}(f;x)|| \le \frac{3}{2}\omega_1\left(\frac{\sqrt{n+2(1-\alpha)}}{n}\right).$$

**Theorem 2.2.8.** *[34] Let be $f\ inC[0,1]$. If $f(x)$ is increasing (or decreasing ) by $[0,1]$ for $0 \le \alpha \le 1$, then the operator $\alpha$-Bernstein is often increasing (respectively decreasing).*

**Theorem 2.2.9.** *[34] Let $f \in C[0,1]$. If $f(x)$ is convex on $[0,1]$, then also the operator $\alpha$-Bernstein is convex for any $0 \le \alpha \le 1$.*

The Chlodovsky variant of the operators $T_{n,\alpha}$ was studied in [116]. It will be presented in Chapter 3.

## 2.3 Operators obtained by iteration

In this paragraph we will construct a new class of operators that are obtained by iteration. In this construction we use a method of modifying the $alpha$ - Bernstein operators, making at each step a convenient choice of the $\alpha$ parameter. The main goal is to obtain operators that have the property of preserving higher order convexity. The results of this section are included in the paper [97].

For $0\ leqr < n$ we consider the operator

$$T_n^r(f)(x) = \sum_{i=0}^{n-r} p_{n-r,i}(x)F_{n,i}^r(f),\ f:[0,1]\to\mathbb{R},\ x\in[0,1], \tag{2.16}$$

where the functionals $F^r_{n,i}$ are defined recursively by $F^0_{n,i}(f) = f\left(\frac{i}{n}\right)$, $0 \le i \le n$ and for $r \ge 1$:

$$F^r_{n,i}(f) = \left(1 - \frac{i}{n-r}\right) F^{r-1}_{n,i}(f) + \frac{i}{n-r} F^{r-1}_{n,i+1}(f),\ 0 \le i \le n-r. \tag{2.17}$$

We observe that for $r = 0$, $T^r_n$ coincides with the Bernstein operator, $B_n$. Also, the operator $T^1_n$ can be compared by operators $T_{n,\alpha}$, define on $T_{n,\alpha} = \alpha B_n + (1-\alpha)T^1_n$, pentru $\alpha \in [0,1]$, introduce by Chen in [34].

For the $T^r_n$ operators we will study in this section the explicit representation, the moments, the estimates of the degree of approximation using moduli of continuity, the Voronoskaja type theorem, the preservation of the higher order convexity and the simultaneous approximation. There is a partial analogy between the $T^r_n$ operators and the iteration of Bernstein operators: $(B_n)^r := B_n \circ \cdots \circ B_n$, ($r$ times).

### 2.3.1 Basic identities

**Lemma 2.3.1.** *For integers $0 \le r < n$, $0 \le i \le n-r$ one has:*

*i)* $F^r_{n,i}(e_0) = 1$,

*ii)* $F^r_{n,i}(e_1) = \frac{i}{n-r}$.

**Corollary 2.3.1.** *For integers $0 \le r < n$, and $x \in [0,1]$, the following relations are true:*

*i)* $T^r_n(e_0)(x) = 1$,

*ii)* $T^r_n(e_1)(x) = x$.

For $a \in \mathbb{R}$, and $n \in \mathbb{N}_0$ we note with $(a)_n$ the Pochhammer symbol, define by $(a)_0 = 1$ and $(a)_n = a(a+1)\dots(a+n-1)$, for $n \ge 1$.

For $n, r, i, k \in \mathbb{N}_0$, $0 \le r \le n$, $0 \le i \le n-r$, $0 \le k \le r$ definim

$$c_{n,r,i,k} = \binom{r}{k}(n-i-r)_{r-k}(i)_k. \tag{2.18}$$

**Lemma 2.3.2.** *For $f \in C[0,1]$, $n \in \mathbb{N}$, $r \in \mathbb{N}_0$, $0 \le r < n$, $0 \le i \le n-r$, one has*

$$F^r_{n,i}(f) = \frac{1}{(n-r)_r}\sum_{k=0}^{r} c_{n,r,i,k} f\left(\frac{i+k}{n}\right). \tag{2.19}$$

**Remark 2.3.1.** From Lemma 2.3.2 it results

$$T^{n-1}_n(f)(x) = (1-x)f(0) + xf(1),\ f : [0,1] \to \mathbb{R},\ n \in \mathbb{N},\ x \in [0,1].$$

This relation prove that operators $T^r_n$ make the connection between the operators $B_n$ and $B_1$, similar to the connection made by $(B_n)^r$, for $r = 1$ and limit wher $r \to \infty$.

For any $n \in \mathbb{N}$ we consider the operator

$$G_n(f)(t) = (1-t)f\left(\frac{n-1}{n}t\right) + tf\left(\frac{n-1}{n}t + \frac{1}{n}\right),\ f \in C[0,1],\ t \in [0,1]. \tag{2.20}$$

**Lemma 2.3.3.** *For $1 \le r < n$ and $f \in C[0,1]$ it holds*

$$T^r_n(f)(x) = (T^{r-1}_{n-1} \circ G_n)(f)(x),\ x \in [0,1]. \tag{2.21}$$

**Corollary 2.3.2.** *For integers $0 \le r < n$ it exists the representation*

$$T^r_n = B_{n-r} \circ G_{n-r+1} \circ G_{n-r+2} \circ \ldots \circ G_n. \tag{2.22}$$

### 2.3.2 The moments

**Lemma 2.3.4.** *For $n \in \mathbb{N}$, $p \in \mathbb{N}$ one has*

$$G_n((e_1 - xe_0)^p)(t) = \sum_{j=0}^{p} (t-x)^j d_{n,p,j}(x),\ t, x \in [0,1], \tag{2.23}$$

*where*

$$\begin{aligned} d_{n,p,j}(x) &= \frac{1}{n^p}\binom{p}{j}(n-1)^j\Big[(1-x)(-x)^{p-j} + x(1-x)^{p-j}\Big] \\ &+ \frac{1}{n^p}\binom{p}{j-1}(n-1)^{j-1}\Big[x(-x)^{p-j} + (1-x)(1-x)^{p-j}\Big]. \end{aligned}$$

We define moments of order $p$ of the operators $T_n^r$, by

$$M^p[T_n^r](x) = T_n^r((e_1 - xe_0)^p)(x),\ 0 \le r < n,\ p \ge 0,\ x \in [0,1]. \tag{2.24}$$

From Lemma **??** and Lemma **??** we have the following recurrence relation

**Corollary 2.3.3.**

$$M^p[T_n^r](x) = \sum_{j=0}^{p} d_{n,p,j}(x) M^j[T_{n-1}^{r-1}](x),\ 1 \le r < n,\ p \ge 0,\ x \in [0,1]. \tag{2.25}$$

**Lemma 2.3.5.** *One has, for $x \in [0,1]$, $0 \le r < n$:*

$$M^0[T_n^r](x) = 1; \tag{2.26}$$

$$M^1[T_n^r](x) = 0; \tag{2.27}$$

$$M^2[T_n^r](x) = \frac{n+r+1}{n(n-r+1)} x(1-x); \tag{2.28}$$

$$M^3[T_n^r](x) = \frac{n^2+4nr+3n+r^2+3r+2}{n^2(n-r+1)(n-r+2)} x(1-x)(1-2x); \tag{2.29}$$

$$M^4[T_n^r](x) = x(1-x)a_{n,r}(x),\ cu\ |a_{n,r}(x)| \le C_r \cdot \frac{1}{n^2} \tag{2.30}$$

*where $C_r$ is independent of $n \in \mathbb{N}$, si $x \in [0,1]$.*

**Lemma 2.3.6.** *For integers $n, r, p$, with $n > r + p$ we have the representation*

$$T_n^r(e_p)(x) = \left(\frac{n-r}{n}\right)^p B_{n-r}(e_p)(x) + R_{n,p,r}(x), \tag{2.31}$$

*where $R_{n,p}(x)$ is a polynomial with degree equal or lower than $p$ having all coefficients of type $\mathrm{O}\left(\frac{1}{n}\right)$, depending on $p$ and $r$.*

### 2.3.3 Estimates of the degree of approximation by the operators $T_n^r$

In this section we deduce the estimates of the approximation order using the moduli of continuity of first order, the usual moduli of continuity of second order and the second order moduli of Ditzian-Totik.

**Theorem 2.3.1.** *For $f \in C[0,1]$, $x \in [0,1]$ and intergers $0 \le r < n$ the following estimations are true:*

$$|T_n^r(f)(x) - f(x)| \le 2\omega_1 \left(f, \sqrt{\frac{(n+r+1)x(1-x)}{n(n-r+1)}}\right), \tag{2.32}$$

$$|T_n^r(f)(x) - f(x)| \le \frac{1}{2}\sqrt{\frac{(n+r+1)x(1-x)}{n(n-r+1)}}\omega_1 \left(f', 2\sqrt{\frac{(n+r+1)x(1-x)}{n(n-r+1)}}\right), \tag{2.33}$$

$$|T_n^r(f)(x) - f(x)| \le \frac{3}{2}\omega_2 \left(f, \sqrt{\frac{(n+r+1)x(1-x)}{n(n-r+1)}}\right), \tag{2.34}$$

$$|T_n^r(f)(x) - f(x)| \le \frac{5}{2}\omega_2^{\varphi} \left(f, \sqrt{\frac{n+r+1}{n(n-r+1)}}\right), \tag{2.35}$$

*where, additionally, in inequality (**??**) we assume that $f \in C^1[0,1]$, in the inequality (2.34) we assume that $2\sqrt{\frac{(n+r+1)x(1-x)}{n(n-r+1)}} \le 1$ and in the inequality (2.35) we assume that $2\sqrt{\frac{n+r+1}{n(n-r+1)}} \le 1$.*

**Corollary 2.3.4.** *For all $f \in C[0,1]$ and the integer $r \ge 0$ it holds:*

$$\lim_{n\to\infty} \|T_n^r(f) - f\| = 0, \tag{2.36}$$

We will give a quantitative version of the Voronoskaya theorem. For this we will use the smallest concave majorant of the first continuity module, given for the function $f \in B[a,b]$ and $h > 0$ from

$$\tilde{\omega}_1(f,h) = \begin{cases} \sup\limits_{\substack{0\le x\le h\le y\le b\\ x\ne y}} \frac{(h-x)\omega_1(f,y)+(y-h)\omega_1(f,x)}{y-x}, & 0 < h \le b-a \\ \omega_1(f,1), & h > b-a. \end{cases} \tag{2.37}$$

**Theorem 2.3.2.** *If $f \in C^2[0,1]$, $r \ge 0$ is an integer and $x \in [0,1]$, then we have*

$$\left|T_n^r(f)(x) - f(x) - \frac{1}{2}\cdot\frac{(n+r+1)x(1-x)}{n(n-r+1)}\cdot f''(x)\right| \tag{2.38}$$

$$\le \tilde{C}_r \frac{x(1-x)}{n}\tilde{\omega}_1\left(f'', \frac{1}{\sqrt{n}}\right), \tag{2.39}$$

*where $\tilde{C}_r > 0$ is an independent constant of $f$, $n$ and $x$.*

### 2.3.4 Higher order convexity. Simultaneous approximation

We remind that we denote by D the derivation operator and by $D^s := D\circ D\circ D\circ\cdots\circ D$, ($s$ -times), the derivative operator of order $s$. If $f \in C^{s+1}(I)$, then $f$ is convex of order $s$ if and only if $D^{s+1}f(x) \ge 0$, for any $x \in I$. An operator that transforms any $s$-convex function into a $s$-convex function is called a $s$ convex operator.

**Lemma 2.3.7.** *For $f \in C[0,1]$, and integers $0 \le r < n$, $0 \le s < n-r$ we have*

$$D^sT_n^r(f)(x) = \frac{(n-r-s+1)_s}{(n-r)_r}\sum_{i=0}^{n-r-s} p_{n-r-s,i}(x)\sum_{k=0}^{r} c_{n+s,r,i+s,k}\Delta_{\frac{1}{n}}^s f\left(\frac{i+k}{n}\right). \tag{2.40}$$

**Theorem 2.3.3.** *Let the whole $n, r$ so that $n > r$. Then the operator $T_n^r$ is convex of order $s$ for any integer $s \geq -1$ so that $n > r + s$.*

With this fact we deduce the property of the simultaneous approximation of the operators $T_n^r$.

**Theorem 2.3.4.** *For integers $0 \leq r < n$ and $0 \leq s < n - r$ we have*

$$\lim_{n \to \infty} \|(D^s \circ T_n^r)(f) - D^s f\| = 0 \tag{2.41}$$

# Chapter 3

# Approximation operators for non-compact intervals

The chapter presents two types of approximation on non-compact intervals, these being the uniform approximation on compact intervals and the weighted approximation.

First we introduce the following basic notions that we will use.

We call weight on the interval $[0,\infty)$, a function $\rho \in C[0,\infty)$ with the property that $\rho(x) > 0,\ x \in [0,\infty)$ and $\lim_{x\to\infty} \rho(x) = 0$.

We define the following function spaces:

- $UC_b(I)$ - the space of real, uniformly continuous and bounded functions defined on I, where I is a real interval
- $B_\rho[0,\infty) = \{f : [0,\infty) \to \mathbb{R} | \exists M_f > 0 \text{ astfel incat } \rho(x)f(x) \leq M_f\ (\forall)x \in [0,\infty)\}$ where $M_f$ is a constant depending on $f$.
- $C_\rho^b[0,\infty) = C[0,\infty) \cap B_\rho[0,\infty)$
- $C_\rho^*[0,\infty) = \{f \in C[0,\infty) |\ \exists\ \lim_{x\to\infty} f(x)\rho(x) \in \mathbb{R}\}$
- $C_\rho^0[0,\infty) = \{f \in C[0,\infty) |\ \exists\ \lim_{x\to\infty} f(x)\rho(x) = 0 \in \mathbb{R}\}$

If $\rho(x) = \frac{1}{1+x^N}$ we note $C_\rho^*[0,\infty)$ with $C_N^*[0,\infty)$.

We can easily see that $C_\rho^0[0,\infty) \subset C_\rho^*[0,\infty) \subset C_\rho^b[0,\infty) \subset B_\rho[0,\infty)$.

On $B_\rho[0,\infty)$ we define the norm $||f||_\rho$ by

$$||f||_\rho = \sup_{x\in[0,\infty)} \rho(x)|f(x).|$$

In relation to this norm $B_\rho[0,\infty)$ is a Banach space.

Also, for a compact set $K$ we define the supremum norm of the function $f$ on the set K by

$$||f||_K = \sup_{x\in K} |f(x)|.$$

## 3.1 General results in the theory of non-compact approximation of linear and positive operators

We begin with an example given by Z. Ditzian [45] which shows that Korovkin's theorem is not true for all sequences of linear and positive operators applied to continuous functions defined on a non-compact interval, even if we restrict to a compact interval the functions obtained as images.

We consider the sequence of linear and positive operators $(L_n)_n,\ L_n : C^0_\rho[0,\infty) \to C[0,1]$, where $\rho(x) = e^{-x}$, define by

$$L_n(f,x) = B_n(f|_{[0,1]};x) + f(n)e^{-n},\ x \in [0,1],\ f \in C^0_\rho[0,\infty),$$

also $B_n$ is Bernstein operator.
Then $L_n(e_k;x) \to x^k$ for $k = 0,1,2$ uniformly on $[0,1]$, but $L_n(e^t;x) \to e^x + 1$ uniformly on $[0,1]$.

In order to obtain the extension of Korovkin's theorem in the case of non-compact intervals, additional conditions are necessary.

In [7], F. Altomare presented general Korovkin-type results in a more abstract context of metric spaces. The most representative are given by the following theorems. First we consider the metric space $(X,d)$. Note with

$$d_x(y) := d(x,y),\ x,\ y \in X.$$

**Theorem 3.1.1.** *[7] Let $(X,d)$ a metric space and be $E$ a vector sublate of his $\mathcal{F}(X)$ which contains constant functions and all functions $d_x^2 (x \in X)$ where $\mathcal{F}(X)$ is the space of real function on $X$. Let $(L_n)_{n\geq 1}$ a sequence of linear and positive operators from $E$ to $\mathcal{F}(X)$ and let $Y$ a set of $X$ such that*

*(i)* $\lim_{n\to\infty} L_n(e_0) = 1$ *uniformly on $Y$*

*(i)* $\lim_{n\to\infty} L_n(d_x^2;x) = 0$ *uniformly with $x \in Y$.*

*Then for any $f \in E \cap UC_b(X)$*

$$\lim_{n\to\infty} L_n(f) = f \textit{ uniformly on } Y$$

*where $UC_b(X)$ is the space of uniformly continuous and bounded functions on $X$.*

**Theorem 3.1.2.** *[7] Let $(X,d)$ a local compact metric space and let $E$ be a vector sublate of $\mathcal{F}(X)$ which contains the constant function 1 and all functions $d_x^2 (x \in X)$. Let $L_n : E \to \mathcal{F}$ a sequence of linear and positive operators so that*

*(i)* $\lim_{n\to\infty} L_n(e_0) = 1$ *uniformly on the compact subsets of $X$*

*(i)* $\lim_{n\to\infty} L_n(d_x^2;x) = 0$ *uniformly on the compact subsets of $X$*

*where $n \geq 1$. Then for all $f \in E \cap C_b(X)$*

$$\lim_{n\to\infty} L_n(f) = f \textit{ uniformly on the compact subsets of } X$$

*where $UC_b(X)$ is the space of uniformly continuous and bounded functions on $X$.*

J. Bustamante presents in [27] some direct theorems for strings of linear and positive operators in weighted spaces. We consider $m \in \mathbb{N}$ fixed and the weight

$$\rho(x) = \rho_m(x) = (1+x)^{-m},\ x \in [0,\infty).$$

For $a \in \mathbb{N},\ b > 0,\ c \geq 0$ we note

$$\varphi(x) = \sqrt{(1+ax)(bx+c)}$$

For normal estimates we use the following weighted moduli for any $f \in C^*_\rho[0,\infty)$ cu

$$\omega_2^{\varphi}(f,t)_\rho = \sup_{h\in(0,t]} \sup_{x\in I(\varphi,h)} |\rho(x)\Delta^2_{h\varphi(x)}f(x)| \tag{3.1}$$

where $I(\varphi,h) = \{x > 0 : h\varphi \leq x\}$. In [28], M. Becker introduced the following moduli of smoothness, for $f \in C_\rho[0,\infty)$ and $h > 0$

$$\omega_2(f,t)_\rho = \sup_{o<h\leq t} \sup_{x\geq 0} \rho(x)|f(x+2h) - 2f(x+h) + f(x)| \tag{3.2}$$

We present below with an equivalent formula the result obtained by J. Bustamante in [27]:

**Theorem 3.1.3.** *[27] Let $m$ a fixed natural number and be $\rho(x) = (1+x)^{-m}$. For $a \in \mathbb{N}$ and $b,\ c \in \mathbb{R}$ with $b > 0$ and $c > 0$. We consider $\varphi(x) = \sqrt{x(1+ax)}$ and a linear and positive operator $L_n : C^*_\rho[0,\infty) \to C[0,\infty)$ having the following properties:*

*(i)* $L_n(e_0) = e_0$ *and* $L_n(e_1) = e_1$

*(ii) it exists a constant $C_1$ and a sequence $(\alpha_n)_n$ such that*

$$L_n\left((e_1 - e_0x)^2, x\right) \leq C_1\alpha_n\varphi^2(x),\ \ x \geq 0.$$

*(iii) it exists a constant $C_2 = C_2(m)$ such that for any $n \in \mathbb{N}$*

$$L_n\left((e_0 + e_1)^m, x\right) \leq C_2(1+x)^m,\ \ x \geq 0.$$

*(iv) it exists a constant $C_3 = C_3(m)$ such that for any $n \in \mathbb{N}$*

$$\rho(x)L_n\left(\frac{(e_1 - e_0x)^2}{\rho(x)}; x\right) \leq C_3\alpha_n\varphi^2(x),\ \ x \geq 0$$

*Then there is a constant $C$ so that for any $f \in C_\rho[0,\infty)$ and $n \in \mathbb{N}$ such that $\alpha_n \leq \frac{1}{2\sqrt{2+a}}$*

$$||f - L_n(f)||_\rho \leq C\omega^2_\varphi\left(f, \sqrt{\alpha_n}\right)_\rho, \quad f \in C^*_\rho[0,\infty)$$

*where $\omega^2_\varphi(f,t)_\rho$ is the moduli from (3.1).*

**Theorem 3.1.4.** *[27] We assume the conditions in Theorem 3.1.3, except that $L_n(e_1; x) = e_1$. Then*

*i) there is a constant $C$ so that for any $f \in C^*_\rho[0,\infty)$ and $n \in \mathbb{N}$ such that $\alpha_n \leq \frac{1}{2\sqrt{2+a}}$*

$$||f - L_n(f)||_\rho \leq C\omega^1_\varphi\left(f, \sqrt{\alpha_n}\right)_\rho$$

*where*

$$\omega^1_\varphi(f,t)_\rho = \sup_{h\in(0,t]} \sup_{2t\geq h\varphi(x)} \left|\rho(x)\left(f\left(x + \frac{h}{2}\varphi(x)\right) - f\left(x - \frac{h}{2}\varphi(x)\right)\right)\right|$$

.

*ii) there is a constant $C$ so that for any $f \in C^*_\rho[0,\infty)$ and $n \in \mathbb{N}$*

$$\rho(x)|f(x) - L_n(f;x)| \leq C\omega_1\left(f, \sqrt{\alpha_n}\right)_\rho,\ \ x \geq 0$$

*where*

$$\omega_1(f,t)_\rho = \sup_{0\leq h\leq t} \sup_{x\geq 0} \rho(x)|f(x+h) - f(x)|$$

.

R. Păltănea presented in [89] estimates with explicit constants of the degree of approximation by linear and positive operators on the interval $[0,\infty)$ using weighted moduli of continuity.

We start by considering the following:

**Definition 3.1.1.** *[89] A function $\varphi \in C[0,\infty)$ is named admissible if it verifies the following conditions:*

*(i)* $\varphi(t) > 0,\ (\forall)t \in (0,\infty)$.

*(ii)* $\frac{1}{\varphi}$ *is convex on* $(0,\infty)$.

*(iii) For $x > 0$ we have*

$$\int_{0+0}^{x} \frac{dt}{\varphi(t)} < \infty;\ for(\forall)x > 0. \tag{3.3}$$

*(iv) One has*

$$\int_{0+0}^{x} \frac{dt}{\varphi(t)} = \infty. \tag{3.4}$$

In this definition we use the improper Riemann integral. Using a permissible $\varphi$ function, we introduce the following first-order weighted moduli:

**Definition 3.1.2.** *[89] For $f \in \mathcal{F}[0,\infty)$ and $h > 0$ we have*

$$\omega^{\varphi}(f,h) = \sup\left\{|f(v) - f(u)|,\ u,v \in [0,\infty),\ |u-v| \le h\varphi\left(\frac{u+v}{2}\right)\right\} \tag{3.5}$$

We suppose that in definition that supremum can be equal to $\infty$.

For an admissible function $\varphi$ we attach the following function:

$$\theta(x) = \int_{0+0}^{x} \frac{dt}{\varphi(t)},\ x \in (0,\infty) \tag{3.6}$$

**Theorem 3.1.5.** *[89] Let $E$ be a linear subspace of $\mathcal{F}[0,\infty)$ and let $F : W \to \mathbb{R}$ a positive linear functional. Let $x \in [0,\infty)$ and $\varphi$ an admissible function. We assume that $(\theta - \theta(x)e_0)^2 \in W$ and $e_0 \in W$. Then for any $f \in W$ and any $h > 0$ we have*

$$\begin{aligned} |F(f) - f(x)| \le &|f(x)| \cdot |F(e_0) - 1| \\ &+ \left(F(e_0) + h^{-2}F\left((\theta - \theta(x)\cdot e_0)^2; x\right)\right)\omega^{\varphi}(f,h). \end{aligned} \tag{3.7}$$

In the case of $\varphi = e_0$ we have $\theta = e_1$ and the relation (3.7) becomes the well-known estimate of Mond.

We present an estimate for the weight $\varphi(x) = \frac{\sqrt{x}}{1+x^n},\ n \in \mathbb{N},\ n \ge 2$.

**Theorem 3.1.6.** *[89] Let $W \subset \mathcal{F}[0,\infty)$ a linear subspace such that $\prod_{2n} \in W$. If $L : W \to F[0,\infty)$ is a linear and positive operator, then for any $f \in W,\ x \in [0,\infty)$ and $h > 0$ we have*

$$\begin{aligned} |L(f;x) - f(x)| \le &|f(x)| \cdot |L(e_0;x) - 1| + \\ &\left(L(e_0;x) + \frac{4}{h^2 x} L\left((e_1 - xe_0)^2(2e_0 + x^{2n}e_0 + e_{2n}); x\right)\right) \cdot \omega^{\varphi}(f,h). \end{aligned} \tag{3.8}$$

## 3.2 General estimates of the weighted approximation on non-compact intervals using classical moduli of continuity

In this subchapter we present some results obtained for the weighted approximation on the interval $[0,\infty)$ by linear and positive operators. Special attention is given to the weights 1 and $1/(1+x^2)$. Applications to these results are given for operators Sz 'asz-Mirakjan and operators Baskakov. We fix a weight function $\rho$ defined on the interval $[0,\infty)$, that is a continuous and strictly positive function on this interval. The results of this section are included in the paper [94].

To estimate the approximation of the function in the space $C_\rho^*[0,\infty)$, through a series of linear and positive operators defined on this space we can apply different weighted moduli of the first order, see [53], [27], [77], [128], [66], [89] and many others.
We will consider here another approach that leads to estimates with classical moduli. Thus, one method consists in reducing the approximation problem on the space $C_\rho^*[0,\infty)$ to the approximation problem on the space $C[0,1]$ using a transformation. This last method of compaction for the convergence problem was used by Bustamante in [25]. We are interested in obtaining the quantitative results of the degree of approximation using this transformation.

For this purpose, we note

$$\psi(y) = \frac{y}{1-y}, \quad y \in [0,1). \tag{3.9}$$

The function $\psi : [0,1) \to [0,\infty)$ it is a bijective with inverse function

$$\psi^{-1}(x) = \frac{x}{1+x}, \quad x \in [0,\infty). \tag{3.10}$$

We consider the linear operator $\Phi : C_\rho^*[0,\infty) \to C[0,1]$ define by

$$\Phi(f,y) = \begin{cases} \rho(\psi(y))f(\psi(y)), & \text{if } y \in [0,1) \\ \lim_{x\to\infty} \rho(x)f(x), & \text{if } y = 1 \end{cases} \quad f \in C_\rho^*[0,\infty) \tag{3.11}$$

The operator $\Phi$ admits the inverse operator $\Phi^{-1} : C[0,1] \to C_\rho^*[0,\infty)$ given by

$$\Phi^{-1}(g,x) = \frac{g(\psi^{-1}(x))}{\rho(x)},\ g \in C[0,1],\ x \in [0,\infty). \tag{3.12}$$

Let $L : C_\rho^*[0,\infty) \to C_\rho^*[0,\infty)$ a linear and positive operator. We consider the linear operator $L^\Phi : C[0,1] \to C[0,1]$, given by:

$$L^\Phi(g) = (\Phi \circ L \circ \Phi^{-1})(g),\ g \in C[0,1], \tag{3.13}$$

which obviously is also a positive operator that satisfies the condition $\|f - L(f)\|_\rho = \|\Phi f - L^\Phi(\Phi f)\|_\infty$, for any $f \in C_\rho^*[0,\infty)$. Using this transformation study weighted approximations through the string of operators $(L_n)_n$, $L_n : C_{\rho,\infty}[0,\infty) \to C_{\rho,\infty}[0,\infty)$ it is reduced to the uniform approximation by the sequences $(L_n^\Phi)_n$, on the simplest space $C[0,1]$.

There are many estimates the moments and with different first and second order moduli of continuity. See [87]. In the following we will use only two general estimates, which are given in the general context of arbitrary intervals to apply them in a double way: directly and by the transformation that follows.

**Theorem 3.2.1.** *Let $\rho$ strictly positive continuous weight over the interval $[0,\infty)$. Let $L : C^*_\rho[0,\infty) \to C^*_\rho[0,\infty)$ a linear and positive operator and $f \in C^*_\rho[0,\infty)$. For any $x \in [0,\infty)$ and $h > 0$ we have*

$$\begin{aligned} |L(f,x)-f(x)| &\leq |f(x)|\Big|\rho(x)L\Big(\frac{e_0}{\rho},x\Big)-1\Big| \\ &+\Big[L\Big(\frac{e_0}{\rho},x\Big)+\frac{1}{h^2}L_n\Big(\frac{(\psi^{-1}-\psi^{-1}(x)e_0)^2}{\rho},x\Big)\Big]\,\omega_1(\Phi f,h). \quad (3.14) \\ |L(f,x)-f(x)| &\leq |f(x)|\Big|\rho(x)L\Big(\frac{e_0}{\rho},x\Big)-1\Big| \\ &+\frac{1}{h}\Big|L\Big(\frac{\psi^{-1}-\psi^{-1}(x)e_0}{\rho},x\Big)\Big|\omega_1(\Phi f,h) \\ &+\Big[L\Big(\frac{e_0}{\rho},x\Big)+\frac{1}{2h^2}L\Big(\frac{(\psi^{-1}-\psi^{-1}(x)e_0)^2}{\rho},x\Big)\Big]\,\omega_2(\Phi f,h). \quad (3.15) \end{aligned}$$

In the following we consider two important weights $\rho(x) = 1$ and $\rho(x) = 1/(1+x^2)$, $x \geq 0$. The approximation in relation to the weight $\rho(x)$ is equivalent to the uniform approximation.

**Theorem 3.2.2.** *Let $\rho(x) = 1$, $x \geq 0$. Let $L : C^*_\rho[0,\infty) \to C^*_\rho[0,\infty)$ a linear and positive operator and $f \in C^*_\rho[0,\infty)$. For any $x \in [0,\infty)$ we have*

$$\begin{aligned} |L(f,x)-f(x)| &\leq |f(x)|\cdot|L(e_0,x)-1| \\ &+(L(e_0,x)+1)\omega_1\left(f\circ\psi,\frac{\sqrt{L(e_1-xe_0)^2,x)}}{1+x}\right) \quad (3.16) \end{aligned}$$

$$\begin{aligned} |L(f,x)-f(x)| &\leq |f(x)|\cdot|L(e_0,x)-1| \\ &+\sqrt{L(e_0,x)}\,\omega_1\left(f\circ\psi,\frac{\sqrt{L(e_1-xe_0)^2,x)}}{1+x}\right) \\ &+\Big(L(e_0,x)+\frac{1}{2}\Big)\omega_2\left(f\circ\psi,\frac{\sqrt{L(e_1-xe_0)^2,x)}}{1+x}\right) \quad (3.17) \end{aligned}$$

**Remark 3.2.1.** *If we apply the estimate from Theorem 1.1.4 and the estimate from Theorem 1.1.8 for $I = [0,\infty)$, $f$ and $x$ for $h = \sqrt{L(e_1-xe_0)^2,x)}$ we obtain*

$$\begin{aligned} |L(f,x)-f(x)| &\leq |f(x)|\cdot|L(e_0,x)-1| \\ &+(L(e_0,x)+1)\omega_1\left(f,\sqrt{L(e_1-xe_0)^2,x)}\right) \quad (3.18) \\ |L(f,x)-f(x)| &\leq |f(x)|\cdot|L(e_0,x)-1| \\ &\frac{|L(e_1-xe_0,x)|}{\sqrt{L(e_1-xe_0)^2,x)}}\omega_1\left(f,\sqrt{L(e_1-xe_0)^2,x)}\right) \\ &+\Big(L(e_0,x)+\frac{1}{2}\Big)\omega_2\left(f,\sqrt{L(e_1-xe_0)^2,x)}\right) \quad (3.19) \end{aligned}$$

**Theorem 3.2.3.** *Let $\rho(x) = \frac{1}{1+x^2}$, $x \geq 0$. Let $L : C^*_\rho[0,\infty) \to C^*_\rho[0,\infty)$ a linear and* *positive operator and* $f \in C^*_\rho[0,\infty)$. *For any* $x \in [0,\infty)$ *we have*

$$\begin{aligned}
\rho(x)|L(f,x)-f(x)| &\le \rho(x)|f(x)| \cdot \left|\frac{L(e_0+e_2,x)}{1+x^2}-1\right| \\
&+ \left[\frac{L(e_0+e_2,x)}{1+x^2}+1\right]\omega_1\left(\Phi(f),\sqrt{\frac{L(e_1-xe_0)^2,x)}{(1+x^2)(1+x)^2}}\right) \qquad (3.20)\\
\rho(x)|L(f,x)-f(x)| &\le \rho(x)|f(x)| \cdot \left|\frac{L(e_0+e_2,x)}{1+x^2}-1\right| \\
&+ \sqrt{\frac{L(e_0+e_2,x)}{1+x^2}}\omega_1\left(\Phi(f),\sqrt{\frac{L(e_1-xe_0)^2,x)}{(1+x^2)(1+x)^2}}\right) \\
&+ \left[\frac{L(e_0+e_2,x)}{1+x^2}+\frac{1}{2}\right]\omega_2\left(\Phi(f),\sqrt{\frac{L(e_1-xe_0)^2,x)}{(1+x^2)(1+x)^2}}\right) \qquad (3.21)
\end{aligned}$$

**Remark 3.2.2.** *From Theorem 1.1.4 and Theorem 1.1.8 for* $I=[0,\infty)$, *for* $h=\sqrt{L(e_1-xe_0)^2,x)}$ *we obtain similar estimates, which can be obtained by multiplying the relations (3.18) and (3.19) by* $\rho(x)$. *And in this case, the relations that are obtained are better in certain situations than the relations (3.20) and (3.21), while in other cases the latter are better.*

## 3.3 Bernstein–Chlodovsky operator

### 3.3.1 General notions

In 1937, I. Chlodovsky introduced a new type of operator in [37]. Thus, for $b>0$ we define the following operator:

$$C_n(f;x) := \sum_{k=0}^{n}\binom{n}{k} f\left(b\cdot\frac{x}{n}\right)\left(\frac{x}{b}\right)^k\left(1-\frac{x}{b}\right)^{n-k}.$$

For $b=1,\ f\in\mathcal{F}[0,1],\ x\in[0,1]$ it becomes Bernstein's operator. The Bernstein operator is transformed from the range $[0,1]$ to the range $[0,b]$.

We are interested in $b$ being as large as possible. In order to obtain the operator Chlodovsky-Bernstein, an interval dilation technique will be applied, which can be represented as follows:

$$C_n(f;x) = B_n\left(f(bt),\frac{x}{b}\right).$$

Chlodovsky used this transformation to approximate functions defined on $[0,\infty)$. For this he chose the variable $b$ as a string $(b_n)_n$ so that $\lim_{n\to\infty} b_n=\infty$. In addition, $\lim_{n\to\infty}\frac{b_n}{n}=0$ is assumed. So Chlodovsky operators have the following form

$$C_n(f;x) = \sum_{k=0}^{n}\binom{n}{k} f\left(b_n\frac{k}{n}\right)\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

In the following theorem we have the point convergence of this operator.

**Theorem 3.3.1.** *[37] If* $f\in C[0,\infty)$ *and the inequality holds*

$$|f(x)| < C\cdot e^{x^p},\ x\ge 0. \qquad (3.22)$$

*where $C$ and $p$ are two arbitrary constants and $x \in [0, \infty)$, and the string $b_n$ verifies the condition*

$$b_n < n^{\frac{1}{p+1+\eta}} \tag{3.23}$$

*where $\eta$ is chosen small enough then*

$$C_n(f; x) \to f(x) \ (n \to \infty)$$

*where $x \in [0, \infty)$ .*

This means that the definition of $C_n$ depends on the increase of $f$. Which makes these operators different from the Szasz-Mirakjian-Favard operators.

For uniform convergence we have the following theorem:

**Theorem 3.3.2.** *[37] If $f \in C[0, \infty)$ and the relation holds*

$$\|f\|_{[0,b_n]} \cdot e^{-\alpha^2 \frac{n}{b_n}} = \mathcal{O}\left(\frac{1}{n}\right) \tag{3.24}$$

*where $\alpha \neq 0$ is finit, then the sequence of polynomials $C_n(f)$ has the property*

$$C_n(f; x) \to f(x) \ \text{când} \ n \to \infty$$

*uniformly for $x \in [a, b]$ for any interval $[a, b] \subset (0, \infty)$.*

For $x \in [0, b_n]$, by using the property of Bernstein operators, we have

i) $C_n(e_0; x) = 1$

ii) $C_n(e_1; x) = x$

iii) $C_n(e_2; x) = x^2 + \frac{x(b_n - x)}{n}$.

iv) $|C_n(f; x) - f(x)| \leq \frac{3}{2}\omega_2\left(f; \sqrt{\frac{x(b_n-x)}{n}}\right)$

### 3.3.2 $\alpha$–Bernstein–Chlodovsky operator

In this sub-chapter, we will introduce the Chlodovsky variant of the $\alpha$ - Bernstein operator which represents a generalization of the $\alpha$-Bernstein operator. We will investigate some elementary properties of this operator and study the approximation properties, including the formula for the asymptotic Voronovskaja type estimation of the operator approximation.

Among the results obtained for the Bernstein-Chlodovsky operators we mention the Voronoskaya type theorem for the derivative of the Kantorovich variant of the Bernstein-Chlodovsky operators, presented by Butzer and Karsli in [24]. Karsli also introduced a variant of the Chlodovsky-Kantorovich operators and a variant of the Chlodovsky-Durrmeyer operators in [71] and [72]. A Chlodovsky variant of the Szasz operators was introduced in [119]. On the other hand, the q-modification of Bernstein-Chlodovsky operators was studied in [70].

The results presented in this sub-chapter have been included in the paper [116].

**Basic properties**

For $\alpha = 1$, the $\alpha$-Bernstein polynomial becomes to the classic Bernstein operator. Our main definition is given below.

**Definition 3.3.1.** *Let $CT_{n,\alpha} : C[0, b_n) \to C[0, b_n)$ be the $\alpha$-Chlodovsky-Bernstein operators, defined by the formula*

$$CT_{n,\alpha}(f; x) := \sum_{i=0}^{n} f\left(\frac{b_n}{n} i\right) p_{n,i}^{\alpha}\left(\frac{x}{b_n}\right) \tag{3.25}$$

*where*

$$p_{n,i}^{(\alpha)}(x) = \left[\binom{n-2}{i}(1-\alpha)x + \binom{n-2}{i-2}(1-\alpha)(1-x) + \binom{n}{i}\alpha x(1-x)\right] \cdot x^{i-1}(1-x)^{n-i-1}, \tag{3.26}$$

*and $\alpha \in \mathbb{R}$, $f \in C[0,\infty)$, $x \in [0, b_n]$ and $(b_n)_{n=1}^{\infty}$ is a positive sequence of real numbers with properties*

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{n} = 0 \tag{3.27}$$

**Remark 3.3.1.** *For $\alpha \in [0, 1]$ and $n \in \mathbb{N}$, the operator $CT_{n,\alpha}(\,\cdot\,; x)$ is pozitiv*

For the family of operators we give some of their properties and results.

**Lemma 3.3.1.** *For all $n \geq 1$, independent of $\alpha$:*

*(i) (Interpolation of functions at the ends of the range) The $\alpha$-Chlodovsky-Bernstein operator for $f$ interpolates $f$ to both ends of the range $[0, b_n]$, so*

$$CT_{n,\alpha}(f; 0) = f(0) \quad \text{and} \quad CT_{n,\alpha}(f; b_n) = f(b_n) \tag{3.28}$$

*(ii) (Linearity) The $\alpha$-Chlodovsky-Bernstein operator is linear, so*

$$CT_{n,\alpha}(\lambda f + \mu g) = \lambda CT_{n,\alpha}(f) + \mu CT_{n,\alpha}(g) \tag{3.29}$$

*for any $f(x)$ and $g(x)$ function defined on $[0,\infty)$, and any real $\lambda$ and $\mu$.*

From Theorem 2.1. from [34] the operator $\alpha$-Chlodovsky-Bernstein can be expressed as follows

**Theorem 3.3.3.** *For $n \in \mathbb{N}$, $x \in [0, b_n]$, $\alpha \in [0, 1]$, $f \in C[0,\infty)$ one has*

$$CT_{n,\alpha}(f; x) = (1-\alpha)\sum_{i=0}^{n-1} g_i \binom{n-1}{i}\left(\frac{x}{b_n}\right)^i \left(1 - \frac{x}{b_n}\right)^{n-i-1} + \alpha \sum_{i=0}^{n} f_i \binom{n}{i}\left(\frac{x}{b_n}\right)^i \left(1 - \frac{x}{b_n}\right)^{n-i} \tag{3.30}$$

*where*

$$g_i = \left(1 - \frac{i}{n-1}\right) f\left(\frac{x}{b_n} i\right) + \frac{i}{n-1} f\left(\frac{x}{b_n}(i+1)\right) \tag{3.31}$$

We will use higher order finite differences to rewrite the operator's shape and to simplify the calculation of the operator's moments. We only need the finite 4th order difference to calculate the 4th order moment of the operator. So we consider $h = \frac{b_n}{n}$ and the polynomial function of degree k $f(x) = x^k$, where $n \geq k$. Then we have

$$\Delta_h^r f(0) = 0 \ for \ r > k \tag{3.32}$$

and

$$\Delta_h^k f\left(\frac{b_n}{n} i\right) = \frac{b_n^k}{n^k} f^{(k)}(\xi_i) = \frac{b_n^k \cdot k!}{n^k}, \quad \xi_i \in \left(\frac{b_n \cdot i}{n}, \frac{b_n \cdot (i+k)}{n}\right) \tag{3.33}$$

The following result is easily obtained:

**Lemma 3.3.2.** *Let* $g(i) = \left(1 - \frac{i}{n-1}\right) f\left(\frac{b_n}{n} i\right) + \frac{i}{n-1} f\left(\frac{b_n}{n}(i+1)\right), \ i \in \{0, 1, \cdots n\}$. *We have*

$$\Delta_1^r g(i) = \left(1 - \frac{i}{n-1}\right) \Delta_1^r f\left(\frac{b_n}{n} i\right) + \frac{i+r}{n-1} \Delta_1^r f\left(\frac{b_n}{n}(i+1)\right) \tag{3.34}$$

*for* $0 \leq r \leq n$.

Now from Theorem 3.1 ([34]), page 6, we have

**Theorem 3.3.4.** *The operator* $\alpha$*-Chlodovsky Bernstein has the following representation in terms of finite differences*

$$CT_{n,\alpha}(f;x) = \sum_{k=0}^{n} \left[(1-\alpha)\binom{n-1}{k} \Delta_1^k g(0) + \alpha \binom{n}{k} \Delta_1^k f(0)\right] x^k \tag{3.35}$$

The Theorem 3.3.4 and Lemma 3.3.2 show that the operator $\alpha$-Chlodovsky Bernstein has the property of preserving the degree of polynomials. In particular, for $f(x) = x^k$ and $n \geq k+1$ it follows that

$$CT_{n,\alpha}(e_k;x) = a_k \left(\frac{x}{b_n}\right)^k + a_{k-1}\left(\frac{x}{b_n}\right)^{k-1} + \ldots + a_1 \left(\frac{x}{b_n}\right) + a_0 \tag{3.36}$$

where

$$a_k = (1-\alpha)\binom{n-1}{k} \Delta_1^k g(0) + \alpha \binom{n}{k} \Delta_h^k f(0) \tag{3.37}$$

**Lemma 3.3.3.** *Let* $CT_{n,\alpha}(f;x)$ *given by (3.25). The first moments of the operator are*

*(i)* $CT_{n,\alpha}(e_0;x) = 1$

*(ii)* $CT_{n,\alpha}(e_1;x) = x$

*(iii)* $CT_{n,\alpha}(e_2;x) = x^2 + \frac{n+2(1-\alpha)}{n^2} x(b_n - x)$

*(iv)* $CT_{n,\alpha}(e_3;x) = x^3 + \frac{3[n+2(1-\alpha)]}{n^2} x^2(b_n - x) + \frac{n+6(1-\alpha)}{n^3} x(b_n - x)(b_n - 2x)$

*(v)* $CT_{n,\alpha}(e_4;x) = x^4 + \frac{6[n+2(1-\alpha)]}{n^2} x^3(b_n - x) + \frac{4[n+6(1-\alpha)]}{n^3} x^2(b_n - x)(b_n - 2x) + \frac{[3n(n-2)+12(n-6)(1-\alpha)]x(b_n-x)+[n+14(}{n^4}$
$x)$

*(vi)* $CT_{n,\alpha}(e_1 - e_0 x;x) = 0;$

*(vii)* $CT_{n,\alpha}\left((e_1-e_0x)^2;x\right)=\frac{n+2(1-\alpha)}{n^2}(b_n-x)x$

*(viii)* $CT_{n,\alpha}\left((e_1-e_0x)^4;x\right)=\frac{[3n(n-2)+12(n-6)(1-\alpha)]x(b_n-x)+[n+14(1-\alpha)]}{n^4}x(b_n-x)$

*where* $n\in\mathbb{N},\ x\in[0,b_n]$.

**Corollary 3.3.1.** *The $\alpha$-Bernstein-Chlodovsky operator reproduces linear functions, therefore*

$$CT_{n,\alpha}(ax+b;x)=ax+b \tag{3.38}$$

*for any real numbers $a$ and $b$.*

**Convergence properties**

For the next result we will use the Theorem 3.1.1.

**Theorem 3.3.5.** *Let $x\in[0,\infty)$, $f\in UC_b[0,\infty)$ and $(b_n)_{n\geq 1}$ defined by (3.27) then for any $K\subset[0,\infty)$ we have*

$$\lim_{n\to\infty}CT_{n,\alpha}(f;x)=f(x) \tag{3.39}$$

*uniformly for $x\in K$.*

To determine the degree of approximation, we will use moduli of continuity of 1st and 2nd order.

**Theorem 3.3.6.** *If $0\leq\alpha\leq 1$, $n\geq 1$ and $x\in[0,b_n]$ then*

$$|CT_{n,\alpha}(f;x)-f(x)|\leq 2\omega_1\left(f;\sqrt{\frac{n+2(1-\alpha)}{n^2}(b_n-x)x}\right) \tag{3.40}$$

**Remark 3.3.2.** *From Theorem 3.3.6 we obtain a quantitative form of Theorem 3.3.5, for $K\subset[0,\infty)$ a compact interval, it results that*

$$x\in K\sqrt{\frac{n+2(1-\alpha)}{n^2}(b_n-x)x}\longrightarrow 0 \text{ when } n\to\infty.$$

*Consequently we have*

$$\lim_{n\to\infty}CT_{n,\alpha}(f,x)=f(x)\quad uniformly\ on\ K\ for\ any\ function\ f\in UC[0,\infty).$$

**Theorem 3.3.7.** *If $0\leq\alpha\leq 1$, $n\geq 1$ and $x\in[0,b_n]$ then*

$$|CT_{n,\alpha}(f;x)-f(x)|\leq \frac{3}{2}\omega_2\left(f;\sqrt{\frac{n+2(1-\alpha)}{n^2}(b_n-x)x}\right) \tag{3.41}$$

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**Theorem 3.3.8.** *Let $f\in C^2[0,\infty)$ and $x\in[0,b_n]$. If $f''$ is uniformly continuous on $[0,\infty)$ and $\lim_{n\to\infty}\omega(f'';\sqrt{\frac{CT_{n,\alpha}((e_1-xe_0)^4;x)}{CT_{n,\alpha}((e_1-xe_0)^2;x)}})=0$, then*

$$\lim_{n\to\infty}\frac{n}{b_n}[CT_{n,\alpha}(f;x)-f(x)]=\frac{1}{2}xf''(x). \tag{3.42}$$

*uniformly on any compact set.*

**Theorem 3.3.9.** *Let $f \in C^2[0,\infty)$, $x \in [0,b_n]$ and $(b_n)_{n=1}^{\infty}$ definied by (3.27). Then*

$$\left|\frac{n}{b_n}\left(CT_{n,\alpha}(f;x)-f(x)\right)-\frac{1}{2}xf''(x)\right| \le |f''(x)|x\left|-\frac{x}{b_n}+\frac{2(1-\alpha)}{n}-\frac{2(1-\alpha)}{nb_n}x\right| + \frac{x}{2}\left(1+\frac{2(1-\alpha)}{n}\right)\tilde{\omega}_1\left(f'';\frac{1}{3}\sqrt{\frac{M_4(x)}{M_2(x)}}\right)$$

*where $M_i(x) = CT_{n,\alpha}\left(|e_1 - xe_0|^i;x\right),\ i \in \mathbb{N}$*

**Remark 3.3.3.** *From Theorem 3.3.9 we get Theorem 3.3.5, if $K \subset [0,\infty)$ is a compact interval, then $\max_{x\in K}\sqrt{\frac{M4_(x)}{M_2(x)}} \longrightarrow 0$ for $n \to \infty$ and $\lim_{n\to\infty} -\frac{x}{b_n}+\frac{2(1-\alpha)}{n}-\frac{2(1-\alpha)}{nb_n}x = 0$. Consequently we have*

$$\lim_{n\to\infty}\frac{n}{b_n}[CT_{n,\alpha}(f;x)-f(x)] = \frac{1}{2}xf''(x) \tag{3.43}$$

*uniformly on $K$ for any compact $K \subset [0,\infty)$ if $f \in C^2[0,\infty),\ x \in K$.*

In the following we will present the most representative operators in the weighted approximation together with their remarkable properties.

## 3.4 Szasz–Favard–Mirakjian operators

**Definition 3.4.1.** *[81] The Szasz–Mirakjian operators can be defined as $S_n : C_2^*[0,\infty) \to C[0,\infty)$*

$$S_n(f;x) = e^{-nx}\sum_{n=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right). \tag{3.44}$$

*where $f \in \mathcal{F}[0,\infty)$ it is chosen so that the series is convergent.*

**Theorem 3.4.1.** *[105] If $f \in C_\rho^*[0,\infty)$, where $\rho(x) = e^{-x}$, then*

$$\lim_{n\to\infty} S_n(f;x) = f(x) \quad uniformly\ on\ [0,\infty).$$

**Theorem 3.4.2.** *[5] The Mirakjian–Favard–Szasz operators have the following properties:*

*1)*

$$\begin{aligned} &S_n(e_0;x) = 1;\\ &S_n(e_1;x) = x;\\ &S_n(e_2;x) = x^2 + \frac{x}{n} \text{where } x \ge 0. \end{aligned}$$

*2) For $f \in C_2^*[0,\infty),\ \lim_{n\to\infty} S_n f = f$ uniformly for any compact $[a,b],\ b > 0$ and*

$$|S_n(f;x) - f(x)| \le 2\omega\left(f, \sqrt{\frac{x}{n}}\right),\ x \in K.$$

*3) For any $f$ derivable on $[0,\infty)$ such that $f' \in C_b[0,\infty)$ one has*

$$|S_n(f;x) - f(x)| \le 2\cdot\sqrt{\frac{x}{n}}\omega\left(f', \sqrt{\frac{x}{n}}\right),\ x \ge 0.$$

**Theorem 3.4.3.** *[5] If $S_n,\ n \geq 1$ are defined by (3.44) then*

1) $S_n^{(r)}(f;x) = n^r e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \Delta_{\frac{1}{n}}^r f\left(\frac{k}{n}\right), \quad r \in \mathbb{N}.$

2) *For $b > 0, (\forall) x \in [0,b],\ (\forall) f \in C^{r+1}[0,\infty)$ such that $f^{(r+1)}$ is bounded on $[0,\infty)$ it holds*

$$|S_n^{(r)}(f;x) - f^{(r)}(x)| \leq \frac{r}{n}\|f^{(r+1)}\| + \frac{1}{\sqrt{n}} K_{n,r}\omega\left(f^{(r+1)}; \frac{1}{\sqrt{n}}\right)$$

*where $\|\cdot\|$ represents the supreme norm on $[0,b]$ and $K_{n,r} = \sqrt{2b + \frac{2r^2}{n}} + b + \frac{r^2}{n\sqrt{n}}$.*

For the uniform approximation we will be in the space $C_b[0,\infty)$. Either the weight function $\varphi,\ \varphi(x) = \sqrt{x},\ x \geq 0$; we use the smoothness moduli

$$\omega_{2,\varphi}(f,\delta) = \sup_{0<h\leq\delta} \|\Delta_{h\varphi}^2 f\|.$$

V. Totik [123] proved the following theorem:

**Theorem 3.4.4.** *For $f \in C_b[0,\infty)$ the following statements are equivalent:*

1) $S_n f - f = o(1)\ (n \to \infty)$.

2) $\omega_{2,\varphi}(f,\delta) = o(1)\ (\delta \to 0^+)$.

3) $f(x + h\sqrt{x}) - f(x) = o(1)\ (h \to 0^+)$ *uniformly on* $[0,\infty)$.

4) $f \circ e_2$ *is uniformly continuous.*

We will show that the results in Section 3.2 can be applied. the Szasz-Favard-Mirakjian operator.

These results were obtained in the paper [94]. Let function $\psi(t) = \frac{t}{1-t},\ t \in [0,1)$.

**Theorem 3.4.5.** *[94] For $\rho(x) = 1,\ x \in [0,\infty)$ and $f \in C_\rho^*[0,\infty)$, it holds:*

$$|S_n(f,x) - f(x)| \leq 2\omega_1\left(f \circ \psi, \sqrt{\frac{x}{n(1+x)^2}}\right),\ x \in [0,\infty),\ n \in \mathbb{N}, \tag{3.45}$$

$$\|S_n f - f\| \leq 2\omega_1\left(f \circ \psi, \frac{1}{2\sqrt{n}}\right),\ n \in \mathbb{N}. \tag{3.46}$$

**Corollary 3.4.1.** *If $f \in C_{e_0}^*[0,\infty)$ then*

$$\lim_{n\to\infty} \|S_n f - f\| = 0$$

**Theorem 3.4.6.** *[94] For $\rho(x) = e_0,\ f \in C_\rho^*[0,\infty)$, and $x \in [0,\infty)$ it holds:*

$$|S_n(f,x) - f(x)| \leq 2\omega_1\left(f, \sqrt{\frac{x}{n}}\right), \tag{3.47}$$

$$|S_n(f,x) - f(x)| \leq \frac{3}{2}\omega_2\left(f, \sqrt{\frac{x}{n}}\right). \tag{3.48}$$

**Remark 3.4.1.** *From the relations (3.47) and (3.48) we cannot obtain the Corollary 3.4.1. We can obtain the uniform convergence of $S_n f$ to $f$ on compact sets from $[0,\infty)$. Moreover, no other choice of the argument $h = h_n(x) > 0$ in Theorem 1.1.4 and Theorem 1.1.8, for $L = S_n$ does not lead to Corollary 3.4.1.*

We consider the weight $\rho(x) = \frac{1}{1+x^2}$, $x \geq 0$.

**Theorem 3.4.7.** *[94] Let* $\rho(x) = \frac{1}{x^2+1}$, $x \in [0, \infty)$ *and let function* $\Phi : C^*_\rho[0, \infty) \to C[0, 1]$*, defined by (3.11). For* $f \in C^*_\rho[0, \infty)$*,* $x \in [0, \infty)$*,* $n \in \mathbb{N}$ *it holds:*

$$\begin{aligned} \rho(x)|S_n(f, x) - f(x)| \quad &\leq \quad \frac{x}{n(1+x^2)}|\rho(x)f(x)| \\ &+ \left(2 + \frac{x}{n(1+x^2)}\right)\omega_1\left(\Phi(f), \sqrt{\frac{x}{n(1+x)^2(1+x^2)}}\right), \end{aligned} \tag{3.49}$$

$$\|S_n f - f\|_\rho \leq \frac{1}{2n}\|f\|_\rho + \left(2 + \frac{1}{2n}\right)\omega_1\left(\Phi(f), \frac{\sqrt{1781}}{100\sqrt{n}}\right). \tag{3.50}$$

**Corollary 3.4.2.** *If* $\rho(x) = \frac{1}{1+x^2}$, $x \in [0, \infty)$ *and* $f \in C^*_\rho[0, \infty)$ *then*

$$\lim_{n\to\infty} \|S_n f - f\|_\rho = 0$$

**Theorem 3.4.8.** *[94] Let* $\rho(x) = e_0/(e_0 + e_2)$*. For any* $f \in C^*_\rho[0, \infty)$ *and* $x \in [0, \infty)$ *it holds:*

$$\rho(x)|S_n(f, x) - f(x)| \quad \leq \quad \frac{1+x}{1+x^2}\omega_1\Big(f, \frac{1}{\sqrt{n}}\Big), \tag{3.51}$$

$$\rho(x)|S_n(f, x) - f(x)| \quad \leq \quad \frac{2+x}{2(1+x^2)}\omega_2\Big(f, \frac{1}{\sqrt{n}}\Big),. \tag{3.52}$$

*Consequently, for any* $n \in \mathbb{N}$ *we have*

$$\|S_n f - f\|\rho \quad \leq \quad \frac{1+\sqrt{2}}{2}\omega_1\Big(f, \frac{1}{\sqrt{n}}\Big), \tag{3.53}$$

$$\rho(x)||S_n f - f\|_\rho \quad \leq \quad \frac{\sqrt{2}}{4}\omega_2\Big(f, \frac{1}{\sqrt{n}}\Big). \tag{3.54}$$

**Remark 3.4.2.** *[94] From the relation (3.53) or the relation (3.54) and the Theorem 3.4.4 we deduce the Corollary 3.4.2. In other words Corollary 3.4.2 can be proved by Theorem 1.1.4 or Theorem 1.1.8. So in the case of the weight* $\rho = e_0/(e_0 + e_2)$ *is a different situation than in the case of the weight* $\rho = e_0$*, see the Observation 3.4.1*

## 3.5 Baskakov operator

**Definition 3.5.1.** *[18] Is named the Baskakov operators the operators* $V_n : C^*_2[0, \infty) \to C[0, \infty)$ *given by the following relation:*

$$BA_n(f; x) = (1 + x)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n}\right) \tag{3.55}$$

**Theorem 3.5.1.** *[5] The Baskakov operators have the following properties:*

*1)* $BA_n(e_0; x) = 1$, $\quad BA_n(e_1; x) = x$, $\quad BA_n(e_2; x) = x^2 + \frac{x(1+x)}{n}$, $x \geq 0$.

*2) For any $f \in C_2^2[0,\infty)$, $\lim_{n\to\infty} BA_n f = f$ uniformly on each compact $[0,b]$, $b>0$ and*

$$|BA_n(f;x)-f(x)| \leq 2\omega\left(f, \sqrt{\frac{x(1+x)}{n}}\right),\ x\in[0,b].$$

*3) For any $f$ differentiable on $[0,\infty)$ such that $f' \in C_b[0,\infty)$*

$$|BA_n(f;x)-f(x)| \leq 2\sqrt{\frac{x(1+x)}{n}}\omega\left(f', \sqrt{\frac{x(1+x)}{n}}\right),\ x\geq 0.$$

**Theorem 3.5.2.** *[5] It holds:*

*(i) If $f$ is convex on $[0,\infty)$ then for any $n\in\mathbb{N}$, $x>0$, $BA_n(f,x) > BA_{n+1}(f;x)$.*

*(ii) If $f$ is concave on $[0,\infty)$ then any $n\in\mathbb{N}$, $x>0$, $BA_n(f,x) < V_{n+1}(f;x)$.*

In the following, we present the results obtained in the paper [94], by applying the general results from section 3.2 to Baskakov's operators.

**Theorem 3.5.3.** *[94] Let the function $\psi$ defined on (3.9). Let be $\rho = e_0$ and $f \in C_\rho^*[0,\infty)$. For $x\geq 0$ and $n\in\mathbb{N}$ the relations hold:*

$$|BA_n(f,x)-f(x)| \leq 2\omega_1\left(f\circ\psi, \sqrt{\frac{x}{n(1+x)}}\right), \tag{3.56}$$

$$|BA_n(f,x)-f(x)| \leq \omega_1\left(f\circ\psi, \sqrt{\frac{x}{n(1+x)}}\right) + \frac{3}{2}\omega_2\left(f\circ\psi, \sqrt{\frac{x}{n(1+x)}}\right), \tag{3.57}$$

$$|BA_n(f,x)-f(x)| \leq 2\omega_1\left(f, \sqrt{\frac{x(1+x)}{n}}\right), \tag{3.58}$$

$$|BA_n(f,x)-f(x)| \leq \frac{3}{2}\omega_2\left(f, \sqrt{\frac{x(1+x)}{n}}\right). \tag{3.59}$$

From relations (3.56) and (3.56) we obtain:

**Corollary 3.5.1.** *[94] Let $\rho = e_0$ and $f \in C_\rho^*[0,\infty)$. For $n\in\mathbb{N}$ it holds:*

$$\|BA_n f - f\| \leq 2\omega_1\left(f\circ\psi, \frac{1}{\sqrt{n}}\right), \tag{3.60}$$

$$\|BA_n f - f\| \leq \frac{3}{2}\omega_2\left(f\circ\psi, \frac{1}{\sqrt{n}}\right), \tag{3.61}$$

*Consequently*

$$\lim_{n\to\infty} \|BA_n f - f\| = 0. \tag{3.62}$$

We will consider the weight $\rho(x) = \frac{1}{1+x^2}$, $x\geq 0$.

**Theorem 3.5.4.** *[94] Let $\rho = e_0/(e_0+e_2)$ and the function $\Phi : C^*_\rho[0,\infty) \to C[0,1]$ defined by (3.11), for any $f \in C^*_\rho[0,\infty)$, any $x \geq 0$ and $n \in \mathbb{N}$ it holds:*

$$\rho(x)|BA_n(f,x) - f(x)| \leq \frac{x(x+1)}{n(x^2+1)}\rho(x)|f(x)| + \left(2 + \frac{x(x+1)}{n(x^2+1)}\right) \cdot \omega_1\left(\Phi(f), \sqrt{\frac{x}{n(x^2+1)(x+1)}}\right), \tag{3.63}$$

$$\rho(x)|BA_n(f,x) - f(x)| \leq \frac{x(x+1)}{n(x^2+1)}|\rho(x)f(x)| + \sqrt{\frac{x(x+1)}{n(x^2+1)}}\omega_1\left(\Phi(f), \sqrt{\frac{x}{n(x^2+1)(x+1)}}\right) + \left(\frac{3}{2} + \frac{x(x+1)}{n(x^2+1)}\right) \cdot \omega_2\left(\Phi(f), \sqrt{\frac{x}{n(x^2+1)(x+1)}}\right), \tag{3.64}$$

$$\rho(x)|BA_n(f,x) - f(x)| \leq \frac{1+x+x^2}{1+x^2}\omega_1\Big(f, \frac{1}{\sqrt{n}}\Big), \tag{3.65}$$

$$\rho(x)|BA_n(f,x) - f(x)| \leq \frac{2+x+x^2}{2(1+x^2)}\omega_2\Big(f, \frac{1}{\sqrt{n}}\Big), \tag{3.66}$$

**Corollary 3.5.2.** *[94] Let $\rho = e_0/(e_0+e_2)$ and $f \in C^*_\rho[0,\infty)$. For $n \in \mathbb{N}$ it holds:*

$$|BA - nf - f\|_\rho \leq \frac{1+\sqrt{2}}{2}\|f\|_\rho + \frac{1+\sqrt{2}}{2}\omega_1\left(\Phi(f), \sqrt{\frac{277}{1000n}}\right), \tag{3.67}$$

$$|BA - nf - f\|_\rho \leq \frac{1+\sqrt{2}}{2}\|f\|_\rho + \sqrt{\frac{1+\sqrt{2}}{2}}\omega_1\left(\Phi(f), \sqrt{\frac{277}{1000n}}\right) + \Big(\frac{3}{2} + \frac{1+\sqrt{2}}{2n}\Big)\omega_2\left(\Phi(f), \sqrt{\frac{277}{1000n}}\right), \tag{3.68}$$

$$|BA_nf - f\|_\rho \leq \frac{3}{2}\omega_1\Big(f, \frac{1}{\sqrt{n}}\Big), \tag{3.69}$$

$$|BA_nf - f\|_\rho \leq \frac{1104}{1000}\omega_2\Big(f, \frac{1}{\sqrt{n}}\Big). \tag{3.70}$$

*Consequently*

$$\lim_{n\to\infty} \|BA_nf - f\|_\rho = 0. \tag{3.71}$$

# Chapter 4

# Semigroups of operators

We will present some results regarding the approximation of semigroups. The results obtained unidimensional will be extended to multidimensional and we will apply them in case of several types of operators.

## 4.1 Introduction

In this section we will briefly illustrate some of the most representative notions in the theory of semigroups of operators and the generalization of their fundamental theorems.

Let $(X, \|\cdot\|)$ be a Banach space. Note by $B(X)$ the space of linear and bounded operators $T : X \to X$, endowed with the norm $\|T\| = \sup_{x\in X,\, \|x\|=1} \|Tx\|$, $T \in B(X)$.

**Definition 4.1.1.** *[126] A family of operators $\{T(t), t \geq 0\}$ , $T(t) \in B(x)$ is named semigroup of operators on $X$ if*

$$T(0) = I;$$
$$T(t+s) = T(t)T(s)$$

*any $t, s \in [0, \infty)$.*

**Definition 4.1.2.** *[126] If the semigroup of operators $\{T(t), t \geq 0\}$ verifies the condition*

$$\lim_{t\to 0+} ||T(t) - I|| = 0$$

*then $\{T(t), t \geq 0\}$ is named the uniformly continuous semigroup of operators.*

**Theorem 4.1.1.** *[126] We can make the following observations:*

1. *If $\{T(t), t \geq 0\}$ is a continuous uniform semigroup, then it exists $\omega \geq 0$ si $M \geq 1$, such that:*

$$||T(t)|| \leq Me^{\omega t}, \quad t \geq 0.$$

2. *If $\{T(t), t \geq 0\}$ is a uniform continuous semigroup, then, for any $x > 0$ we have*

$$\lim_{t\to s} ||T(t) - T(s)|| = 0$$

**Theorem 4.1.2.** *[126] If $\{T(t), t \geq 0\}$ is a uniform continuous semigroup on $X$, then it exists $A \in B(X)$ such that $T(t) = e^{tA}, t \geq 0$, where $e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k$, where $A^k = A \circ \cdots \circ A$ (de $k$ times).*

Since uniformly continuous semigroups are completely determined by the previous theorem, they are of more limited interest.
We will now consider a wider class of semigroup operators.

**Definition 4.1.3.** *Let $\{T(t), t \geq 0\}$, $T(t) \in B(X)$. The semigroup of operators $\{T(t), t \geq 0\}$ is named $C_0$ semigroup of operators if*

$$\lim_{t \to 0+} T(t)x = x, \ (\forall) x \in X$$

*in the sense of the norm $X$.*

It can be seen that any uniformly continuous semigroup is at the same time a $C_0$ semigroup.
Here are some properties of $C_0$ semigroup operators.

**Theorem 4.1.3.** *[126] Let $T(t)$ be a $C_0$ semigroup of operators on $X$. Then it exists the limit*

$$\omega_0 = \lim_{t \to \infty} \frac{\ln ||T(t)||}{t} \in [-\infty, +\infty)$$

*For any number $w > \omega_0$ it exists a constant $M \geq 1$ such that:*

$$||T(t)|| \leq M e^{wt}, t \geq 0.$$

**Corollary 4.1.1.** *[126] Let $\{T(t), t \geq 0\}$ be a $C_0$ semigroup and let $x \in X$. Then the application $t \to T(t)x$ between $[0, \infty)$ and $X$ is continuous.*

**Definition 4.1.4.** *[126] A $C_0$ semigroup $\{T(t), t \geq 0\}$ is named bounded semigroup if it exists $M \geq 1$ such that $||T(t)|| \leq M, t \geq 0$; $\{T(t), t \geq 0\}$ is named semigorup of contractions if $||T(t)|| \leq 1, t \geq 0$.*

Let be a linear operator $A : D(A) \subset X \to X$, where

$$D(A) = \left\{ x \in X : (\exists) \lim_{t \to 0+} \frac{T(t)x - x}{t} \in X \right\}$$

and where the above limit is considered within the meaning of the $X$ norm.
For any $x \in D(A)$ we will consider

$$Ax = \lim_{t \to 0+} \frac{T(t)x - x}{t}$$

We can see that for $x \in D(A)$, the application $t \to T(t)x$ is derivable to the right in 0, and the derivative is equal to $Ax$.
The linear operator A defined above is named *the infinitesimal generator* of the group $\{T(t), t \geq 0\}$.

**Theorem 4.1.4.** *[126] Let $\{T(t), t \geq 0\}$ be a $C_0$ semigroup on $X$ and let $A$ its infinitesimal generator*

1. *If $x \in X$, then we have*

$$\int_0^t T(s)xds \in D(A)$$
$$A\int_0^t T(s)xds = T(t)x - x,\ t \geq 0$$

2. *For $x \in D(A)$ one has $T(t)x \in D(A)$ and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax,\ t \geq 0$$

3. *If $x \in D(A)$ then*

$$T(t)x - x = \int_0^t T(s)Axds = \int_0^t AT(s)xds,\ t \geq 0$$

**Theorem 4.1.5.** *[126] Let $A$ the infinitesimal generator of a $C_0$ semigroup $\{T(t), t \geq 0\}$ of operators on the Banach $X$ space. Then $D(A)$ is dense in $X$, and $A$ is a closed operator.*

**Theorem 4.1.6.** *[126] If $C_0$ semigroups $T(t)$ and $S(t)$ they have the same infinitesimal generator $A$, then $T(t) = S(t),\ t \geq 0$.*

Let $\omega_0 = \inf\{w \in \mathbb{R} : (\exists)M \geq 1, \text{ such that } ||T(t)|| \leq Me^{wt},\ t \geq 0\}$.

**Theorem 4.1.7.** *[126] Let the scalar $\lambda \in \mathbb{C}$ such that $Re\lambda > \omega_0$. Then $\lambda$ is the regular value for $A$, i.e. there is the operator $(\lambda I - A)^{-1} \in B(X)$. Moreover, the bijective, linear and bounded operator $(\lambda I - A)^{-1} : X \to D(A)$ can be described by the formula:*

$$(\lambda I - A)^{-1}x = \int_0^\infty e^{-\lambda t}T(t)xdt,\ x \in X.$$

If $\lambda \in \mathbb{C}$ is regular value, we note:

$$R(\lambda : A)x = \int_0^\infty e^{-\lambda t}T(t)xdt,\ x \in X.$$

**Theorem 4.1.8.** *[126] If we have the semigroup of operators $\{T(t), t \geq 0\}$ having the generator $A$ and $\lambda \in \mathbb{C}$ check the condition $Re\ \lambda > \omega_0$ then is regular value and*

$$R(\lambda : A)x = \int_0^\infty e^{-\lambda t}T(t)xdt,\ x \in X$$

*and*

$$||R(\lambda : A)|| \leq \frac{1}{Re\lambda}.$$

**Theorem 4.1.9.** *[126] Let $A$ the infinitesimal generator of a $C_0$ contraction semigroup and be $\lambda$ real, $\lambda > 0$. Then:*

$$\lim_{\lambda \to \infty} \lambda R(\lambda : A)x = x,\ x \in X.$$

## 4.2 Trotter's theorem

Trotter [124] has shown that a semigroup of operators can be generated by the limiting of the iterations of an operator.
A variant of this result is presented in the following theorem.

**Theorem 4.2.1.** *[124] Let $(X, ||\cdot||)$ a Banach space and let $(L_n)_{n\geq 1}$ be a series of linear operators bounded on $X$. Moreover, we consider a string of positive real numbers $(\rho(n))_{n\geq 1}$ so that $\lim_{n\to 0}\rho(n) = 0$. Suppose there is $M \geq 1$ and $\omega \in \mathbb{R}$ so*

$$||L_n^k|| \leq Me^{\omega\rho(n)k} \tag{4.1}$$

*for any $k, n \geq 1$.*
*Let $(A, D(A))$ be a linear operator on $X$ defined by*

$$A(f) := \lim_{n\to\infty} \frac{L_n(f) - f}{\rho(n)}, \tag{4.2}$$

*for any $f \in D(A)$, where*

$$D(A) := \left\{ g \in X | (\exists) \lim_{n\to\infty} \frac{L_n(g) - g}{\rho(n)} \right\}. \tag{4.3}$$

*Suppose that*

*(a) $D(A)$ is dense on $X$.*

*(b) the range of $(\lambda I - A)(D(A))$ is dense on $X$ for $\lambda > \omega$.*

*Then there is the closure of the operator $(A, D(A))$ is closed and its closure is the generator $C_0$ of a semigroup $(T(t))_{t\geq 0}$, which has the property as for any string $(k(n))_{n\geq 1}$ of natural numbers satisfying $\lim_{n\to\infty} k(n)\rho(n) = t$ we have*

$$T(t)(f) = \lim_{n\to\infty} L_n^{k(n)}(f) \tag{4.4}$$

*for any $f \in X$.*
*Furthermore, $||T(t)|| \leq M\exp^{\omega t}$ for any $t \geq 0$.*

## 4.3 Iterates of the Bernstein operator and the semigroup generated by it

In this subchapter we will focus on the study of the iterations of the Bernstein operator. We will start by presenting their properties.

Iterates of the Bernstein operator have been studied since 1960. The most significant articles we mention are [114], [74], [73], [85], [**?**], [57] and [56]. Some generalizations were given later by Altomare in the articles [8], [9] and [10]. Among the most recent articles we mention [86]. There are also studies of the behavior of Bernstein's iterations presented from a quantitative point of view, as we will see below.

We denote by $B_n, \mathbb{N} \in \mathbb{N}$, the Bernstein operator of rank n given by the relation

$$B_n(f, x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}.$$

where $f \in C[0, 1]$ and $x \in [0, 1]$.

We inductively define his powers $B_n$

$$B_n^0 := Id,\ B_n^1 := B_n\ \cdots\ B_n^{m+1} := B_n \circ B_n^m,\ m \in \mathbb{N}$$

One has:

$$B_1(f,x) = (1-x)f(0) + xf(1).$$

In 1967, Kelinsky and Rivlin [74] proved the following theorem:

**Theorem 4.3.1.** *[74] For any function $f \in C[0,1]$ and any $n \in \mathbb{N}$ one has*

$$\lim_{m\to\infty} ||B_n^m(t) - B_n(f)|| = 0.$$

Among other demonstrations given to this result we mention the demonstration based on the principle of contraction given by I. Rus [104].
Karlin and Ziegert gave a stronger result than Theorem 4.3.1 [73]

**Theorem 4.3.2.** *[73] Let be a string of natural numbers $(m_n)_{n\in\mathbb{N}}$*

*(i) If $\lim_{n\to\infty} \frac{m_n}{n} = 0$, then $\lim_{n\to\infty} ||B_n^{m_n} f - f|| = 0,\ f \in C[0,1]$.*

*(ii) If $\lim_{n\to\infty} \frac{m_n}{n} = \infty$, then $\lim_{n\to\infty} ||B_n^{m_n} f - B_1 f|| = 0,\ f \in C[0,1]$.*

Quantitative estimates of the limits in Theorem 4.3.2 were given by H. Gonska as follows:

**Theorem 4.3.3.** *[57] For $m,\ n \in \mathbb{N},\ f \in C[0,1]$ and the classical moduli of order 2 $\omega_2$ the followings hold:*

*(i)* $|B_n^m(f,x) - f(x)| \le 4 \cdot \omega_2\left(f, \sqrt{\left[1 - \left(1 - \frac{1}{n}\right)^m\right] \cdot x(1-x)}\right)$

*(ii)* $|B_n^m(f,x) - B_1(f,x)| \le 4 \cdot \omega_2\left(f, \sqrt{\left(1 - \frac{1}{n}\right)^m \cdot x(1-x)}\right)$

More generally than the result set out in Theorem 4.3.2, the following result of S. Cooper and S. Waldron can be obtained by using Trotter's theorem:

**Theorem 4.3.4.** *[38] There is a semigroup of operators $\{T(t), t \ge 0\}$, $T(t) : C[0,1] \to C[0,1]$, as for any $t \ge 0$ and any sequence of positive integers $(m_n)_n$, with the property $\frac{m_n}{n} \to t$, we have*

$$\lim_{n\to\infty} B_n^{m_n} f = T(t)f, \quad (\forall) f \in C[0,1]. \tag{4.5}$$

A quantitative estimate of this result was obtained by H. Gonska and I. Raşa as follows:

**Theorem 4.3.5.** *[61] There is a constant $c$ independent of $f \in C[0,1]$, $n \in \mathbb{N}$ and $t \ge 0$ so that for any string $(m_n)_n$, $m_n \in \mathbb{N}$ so that*

$$\lim_{n\to\infty} \frac{m_n}{n} = t \ge 0 \ si\ \delta_n = \max\left\{|\frac{m_n}{n} - t|, \frac{1}{n}\right\} \le 1$$

*it holds*

$$||B_n^{m_n} f - T(t)f|| \le C\left\{\omega_4\left(f, \delta_n^{\frac{1}{4}}\right) + \delta_n^{\frac{1}{2}}\omega_2\left(f, \delta_n^{\frac{1}{4}}\right)\right\}$$

*where*

$$T(t)(f,x) = B_1(f,x) + x(1-x)\int_0^1 G_t(x,y)(f - B_1 f)(y)dy$$

*and the kernel* $G_t$ *is given by*

$$G_t(x,y) = \sum_{k=2}^{\infty} \frac{k(2k-1)}{k-1} e^{-\frac{1}{2}k(k-1)t} \cdot p_{k-2}^{(1,1)}(2x-1) \cdot p_{k-2}^{(1,1)}(2y-1).$$

*Here* $p_{k-2}^{(1,1)}(x)$, $-1 \le x \le 1$ *is the Jacobian polynomial of parameter* $(1,1)$ *in the range* $[-1,1]$ *normalized with the value* $k-1$ *in* $x=1$.

## 4.4 Quantitative estimates for the approximation of the semigroups of operators - the one-dimensional case

### 4.4.1 Quantitative estimates for the limit of the semigroups of positive operators

In this chapter we will give a general quantitative estimate for approximating the iterations of linear and positive operators that preserve the constants at the limit of the semigroup defined by these iterations.

Important results of the operator semigroups generated by the Bernstein operator iterations were given in [74], [115], [61], [36], [80] .

For the semigroups generated by other linear and positive operators, we mention [114], [14], [73], [79], [62].

In recent decades, applications of Trotter's theorem have been considered for the semigroups of operators generated by the iterations of linear and positive operators. For a general reference for the semigroups of operators we mention [10] and [29]. A quantitative version of Trotter's theorem for the semigroup generated by Bernstein operators was first obtained by Gonska and Raşa [61]. In Minea's work, read this method, this method has been improved and applied to general operators who preserve linear functions.

The purpose of this section is to present several general estimates of Trotter's theorem for linear and positive operators that preserve only constants. As an application we will consider Durrmeyer operators. The results were published in [117].
Consider the following series of linear and positive operators $(L_n)_n$, $L_n : C[0,1] \to C[0,1]$, so that $L_n(e_0) = e_0$. Note:

$$m_n^k(x) := L_n((t-x)^k, x), \quad k, n \in \mathbb{N}, \; x \in [0,1].$$
$$M_n^k(x) := L_n(|t-x|^k, x), \quad k, n \in \mathbb{N}, \; x \in [0,1].$$

Suppose that it exits the functions $\varphi_1, \varphi_2, \psi_n^1, \psi_n^2 \in C[0,1]$, $n \in \mathbb{N}$, such that

$$m_n^1(x) = \frac{1}{n}\varphi_1(x) + \psi_n^1(x), \; x \in [0,1], \; n \in \mathbb{N}, \tag{4.6}$$

$$m_n^2(x) = \frac{1}{n}\varphi_2(x) + \psi_n^2(x), \; x \in [0,1], \; n \in \mathbb{N}, \tag{4.7}$$

where

$$\|\psi_n^j\| = o\Big(\frac{1}{n}\Big),\ j = 1, 2,\ (n \to \infty). \tag{4.8}$$

We assume that the operators $L_n$ are convex of order $i$, $i \geq 0$, that is, operators with the property that for any function $f \in C^i[0,1]$, $f^{(i)} \geq 0$, we have $(L_n(f))^{(i)} \geq 0$. We also assume that

$$m_n^4(x) = o(m_n^2(x)),\ x \in [0,1],\ (n \to \infty). \tag{4.9}$$

Then Voronovskaya's theorem assures that

$$\lim_{n\to\infty}(L_n(f,x) - f(x)) = f'(x)\varphi_1(x) + \frac{1}{2}\varphi_2(x)f''(x). \tag{4.10}$$

and the limit is uniform with respect to $x \in [0,1]$, for $f \in C^2[0,1]$.

Let the differential operator $A : C^2[0,1] \to C[0,1]$, given by $A(f)(x) = \varphi_1(x)f'(x) + \frac{1}{2}\varphi_2(x)f''(x)$, $f \in C^2[0,1]$, $x \in [0,1]$. Then the domain $D(A) = C^2[0,1]$ of the operator $A$ is dense in $C[0,1]$.

Trotter's theorem states that there exists $C_0$ semigroup $T(t)$, so

$$\lim_{n\to\infty} L_n^{m_n} f = T(t)f, f \in C[0,1] \tag{4.11}$$

if $\frac{m_n}{n} \to t,\ t \geq 0$.

**Lemma 4.4.1.** *For any $g \in C^4[0,1]$, we have*

$$\left\|L_n g - g - \frac{1}{n}Ag\right\| \leq \frac{1}{6}||M_n^3||\ ||g^{(3)}|| + \|\psi_n^1\| \cdot \|g'\| + \frac{1}{2}\|\psi_n^2\| \cdot \|g''\|.$$

**Lemma 4.4.2.** *For any $g \in C^4[0,1]$, one has:*

$$\begin{aligned}\left\|T\left(\frac{1}{n}\right)g - g - \frac{1}{n}Ag\right\| \quad \leq \quad & \frac{1}{4n^2}\|2\varphi_1\varphi_1' + \varphi_2\varphi_1''\| \cdot \|g'\| \\ & + \frac{1}{8n^2}\|4\varphi_1^2 + 2\varphi_1\varphi_2' + 4\varphi_2\varphi_1' + \varphi_2\varphi_2''\| \cdot \|g''\| \\ & + \frac{1}{4n^2}\|2\varphi_1\varphi_2 + \varphi_2\varphi_2'\| \cdot \|g^{(3)}\| + \frac{1}{8n^2}\|\varphi_2^2\| \cdot \|g^{(4)}\|.\end{aligned}$$

If $(L_n)_n$ is a sequence of linear and positive operators that are convex of any order, then they preserve the degree of polynomials. Thus, $(L_n e_k)^{(k)} \in \mathbb{R},\ n \in \mathbb{N},\ k \in \mathbb{N}_0$. Denote

$$\sigma_k = \frac{1}{k!}\left(L_n e_k\right)^{(k)} \tag{4.12}$$

In [80] the following lemma is demonstrated.

**Lemma 4.4.3.** *If $(L_n)_n$ is a sequence of linear and positive operators that are convex of any order and if $f \in C^k[0,1],\ k \geq 0$ then:*

$$||(L_n^j)^{(k)}|| \leq (\sigma_k)^j ||f^{(k)}||,\ j \geq 0.$$

*where $\sigma_k$ is defined by (4.12).*

Now, the main result of this chapter is given by the following theorem.

**Theorem 4.4.1.** *Let be a sequence of linear and positive operators $(L_n)_n$, $L_n : C[0,1]\ toC[0,1]$, which are convex of any order. Let $f\ inC^4[0,1]$. The following estimate occurs:*

$$\begin{aligned}
||L_n^m f - T(t)f|| \quad \leq \quad & \frac{1-\sigma_1^m}{1-\sigma_1}\|f'\|\left(\|\psi_n^1\| + \frac{1}{4n^2}\|2\varphi_1\varphi_1' + \varphi_2\varphi_1''\|\right) \\
& + \frac{1-\sigma_2^m}{1-\sigma_2}\|f''\|\left(\frac{1}{2}\|\psi_n^2\| + \frac{1}{8n^2}\|4\varphi_1^2 + 2\varphi_1\varphi_2' + 4\varphi_2\varphi_1' + \varphi_2\varphi_2''\|\right) \\
& + \frac{1-\sigma_3^m}{1-\sigma_3}\|f^{(3)}\|\left(\frac{1}{6}||\tilde{M}_n^3\| + \frac{1}{4n^2}\|2\varphi_1\varphi_2 + \varphi_2\varphi_2'\|\right) \\
& + \frac{1-\sigma_4^m}{1-\sigma_4}\frac{1}{8n^2}\|f^{(4)}\|\|\varphi_2^2\| \\
& + \left|\frac{m}{n} - t\right|\left(\|\varphi_1\|\cdot||f'|| + \frac{1}{2}\|\varphi_2\|\cdot||f''||\right), \qquad (4.13)
\end{aligned}$$

*where $\sigma_k$ is defined by (4.12).*

### 4.4.2 Application: Durrmeyer operators

The Durrmeyer operators $D_n : L_1[0,1] \to C[0,1]$ are defined as

$$(D_n f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt,$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, $k = 0, 1, \cdots,$ $x \in [0.1]$. This operators are convex $i \geq 0$. So, if $f \in C^i[0,1]$ and $f^{(i)} \geq 0$ on $[0,1]$, then $(D_n(f))^{(i)} \geq 0$ on $[0,1]$.

We have (from [43]):

$$\begin{aligned}
m_n^1(x) &= \frac{1-2x}{n+2} \\
m_n^2(x) &= \frac{(2n-6)x(1-x)+2}{(n+2)(n+3)} \\
m_n^4(x) &= O\left(\frac{1}{n^2}\right), \text{ uniformly regarding } x \in [0,1],
\end{aligned}$$

Using the notations (4.6) and (4.7) we obtain the relations:

$$\begin{aligned}
&\varphi_1(x) = 1-2x, \quad \psi_n^1(x) = \frac{2(2x-1)}{n(n+2)} \\
&\varphi_2(x) = 2x(1-x), \quad \psi_n^2(x) = -\frac{x(1-x)}{n(n+2)(n+3)} + \frac{2}{(n+2)(n+3)}.
\end{aligned}$$

So,

$$\|\psi_n^1\| \leq \frac{2}{n(n+2)} \leq \frac{2}{n^2}, \quad \|\psi_n^2\| \leq \frac{2}{(n+2)(n+3)} \leq \frac{2}{n^2}.$$

Form [43] it results, by using the notation given by (4.12)

$$\begin{aligned}\sigma_1 &= \frac{n}{n+2}\\ \sigma_2 &= \frac{n(n-1)}{(n+2)(n+3)}\\ \sigma_3 &= \frac{n(n-1)(n-2)}{(n+2)(n+3)(n+4)}\\ \sigma_4 &= \frac{n(n-1)(n-2)(n-3)}{(n+2)(n+3)(n+4)(n+5)}.\end{aligned}$$

We observe that $\sigma_k < 1$, $k = 1, 2, 3, 4$. It results

$$\begin{aligned}\frac{1}{1-\sigma_1} &= \frac{n+2}{(n+2)-n} \leq \frac{3}{2}n\\ \frac{1}{1-\sigma_2} &= \frac{(n+2)(n+3)}{(n+2)(n+3)-n(n-1)} \leq n\\ \frac{1}{1-\sigma_3} &= \frac{(n+2)(n+3)(n+4)}{(n+2)(n+3)(n+4)-n(n-1)(n-2)} \leq n\\ \frac{1}{1-\sigma_4} &= \frac{(n+2)(n+3)(n+4)(n+5)}{(n+2)(n+3)(n+4)(n+5)-n(n-1)(n-2)(n-3)} \leq n.\end{aligned}$$

Since $2\varphi_1(x)\varphi_1'(x) + \varphi_2(x)\varphi_1''(x) = -4(1-2x)$ we deduce

$$\|\psi_n^1\| + \frac{1}{4n^2}\|2\varphi_1\varphi_1' + \varphi_2\varphi_1''\| \leq \frac{2}{n^2} + \frac{1}{n^2} = \frac{3}{n^2}.$$

Since $4\varphi_1^2(x) + 2\varphi_1(x)\varphi_2'(x) + 4\varphi_2(x)\varphi_1'(x) + \varphi_2(x)\varphi_2''(x) = 8 - 56x(1-x)$ we deduce

$$\frac{1}{2}\|\psi_n^2\| + \frac{1}{8n^2}\|4\varphi_1^2 + 2\varphi_1\varphi_2' + 4\varphi_2\varphi_1' + \varphi_2\varphi_2''\| \leq \frac{2}{n^2}.$$

We have $2\varphi_1(x)\varphi_2(x)+\varphi_2(x)\varphi_2'(x) = 8x(1-x)(1-2x)$. Also $M_n^3(x) \leq \sqrt{m_n^2(x)m_n^4(x)}$ and $m_n^2(x) \leq \frac{n+1}{2(n+2}n+3)$ si $m_n^4(x) \leq \frac{2}{n^2}$. We deduce

$$\frac{1}{6}||M_n^3\| + \frac{1}{4n^2}\|2\varphi_1\varphi_2 + \varphi_2\varphi_2'\| \leq \frac{1}{3n\sqrt{n}} + \frac{1}{2n^2}.$$

Since $\varphi_2^2(x) = 4x^2(1-x)^2$ we deduce

$$\frac{1}{8n^2}\|f^{(4)}\|\|\varphi_2^2\| \leq \frac{1}{32n^2}.$$

Finally $\|\varphi_1\| \leq 1$ and $\frac{1}{2}\|\varphi_2\| \leq \frac{1}{4}$.

Using Theorem **??** and the above results obtained, the following estimate was obtained.

**Theorem 4.4.2.** *For any $f \in C^4[0,1]$ and any $0 \leq m \leq n$ one has*

$$\begin{aligned}||D_n^m f - T(t)f|| \leq\ & \frac{9}{2n}\|f'\| + \frac{2}{n}\|f''\|\\ &+ \left(\frac{1}{3\sqrt{n}} + \frac{1}{2n}\right)\|f^{(3)}\| + \frac{1}{32n}\|f^{(4)}\|\\ &+ \left|\frac{m}{n} - t\right|\left(\|f'\| + \frac{1}{4}\|f''\|\right).\end{aligned}$$

## 4.5 Quantitative results for semigroups of operators generated by multidimensional Bernstein operators

As a bibliography of the topic of this section we mention the following references([30]), ([111]), ([84]), ([10]), ([47]), ([11]) and ([6]).

Iterates and semigroups generated by Bernstein operators were studied in [74], [115], [61], [31], [32], [36], [80]. The semigroups generated by the Bernstein multidimensional operators were considered in [31], [32], [79]. For the limits of the group generated by other linear and positive operators we mention [114], [14], [73], [79], [62], [117].

The results from this section were obtained in the paper [96].

### 4.5.1 Auxiliary results for multidimensional Bernstein operators

To define Bernstein multidimensional operators we will consider the following notations.

We consider $d \in \mathbb{N}$ fixed.

To ease multi-index notation $\overline{k} \in \mathbb{N}_0^d$, $\overline{k} = (k_1, \ldots, k_d)$ we denote $|\overline{k}| = k_1 + \ldots + k_d$ and $\overline{k}! = k_1! \ldots k_d!$. For $n \in \mathbb{N}$, if $\overline{k} \in \mathbb{N}_0^d$, $|\overline{k}| \leq n$ we can define $\binom{n}{\overline{k}} = \frac{n!}{\overline{k}!(n-|\overline{k}|)!}$.

We define the $d$-simplex

$$\Delta_d := \{\overline{x} = (x_1, \ldots, x_d) |\ x_i \geq 0,\ (1 \leq i \leq d),\ x_1 + \ldots + x_d \leq 1\}. \tag{4.14}$$

The vectors $\overline{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, $(1 \leq i \leq d)$ form the base of the standard space $\mathbb{R}^d$. If $\overline{x} = (x_1, \ldots, x_d) \in \Delta_d$ we choose $|\overline{x}| = x_1 + \ldots + x_d$. So $|\overline{x}| \leq 1$. Furthermore if, we consider $\overline{k} = (k_1, \ldots, k_d) \in \Lambda_d^n$, then we define $\overline{x}^{\overline{k}} = x_1^{k_1} \ldots x_d^{k_d}$.

With the previous notations we have:

**Definition 4.5.1.** *Is named Bernstein operator on $\Delta_d$ simplex, the operator defined by the following formula*

$$B_n(f, \overline{x}) := \sum_{|\overline{k}| \in \Lambda_d^n} p_{n,\overline{k}}(\overline{x}) f\left(\frac{\overline{k}}{n}\right), \tag{4.15}$$

*where*

$$p_{n,\overline{k}}(\overline{x}) := \binom{n}{\overline{k}} \overline{x}^{\overline{k}} (1 - |\overline{x}|)^{n-|\overline{k}|}. \tag{4.16}$$

*with $\overline{k} \in \mathbb{Z}^d$ such that*

$$p_{n,\overline{k}}(\overline{x}) := 0, \quad \text{if } \exists i, \text{ such that } k_i < 0, \text{ or } |\overline{k}| > n. \tag{4.17}$$

*and $\frac{\overline{k}}{n} = \left(\frac{k_1}{n}, \ldots \frac{k_d}{n}\right)$, $n \in \mathbb{N}$, $f : \Delta_d \to \mathbb{R}$, $\overline{x} \in \Delta_d$.*

In the case $d = 1$ and $\overline{k} = k$, $\overline{x} = x$ we will have the one-dimensional case and we will denote simpler $p_{n,k}(x)$ instead of $p_{n,\overline{k}}(\overline{x})$.

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. Suppsoe that $|\alpha| \geq 1$, where $|\alpha| = \alpha_1 + \ldots + \alpha_d$. If $f \in C^{|\alpha|}(\Delta_d)$ we define

$$\frac{\partial^\alpha f}{\partial \overline{x}^\alpha} := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}. \tag{4.18}$$

If $|\alpha| = 0$, we define $\frac{\partial^\alpha f}{\partial \overline{x}^\alpha} := f$.

For $\alpha \in \mathbb{N}_0^d$ we denote by $C^\alpha(\Delta_d)$, the space of functions $f : \Delta_d \to \mathbb{R}$ which admit the partial derivative $\frac{\partial^\alpha f}{\partial \overline{x}^\alpha}$ continuous on $\Delta_d$. For $1 \le i \le d$ we consider the functions $\pi_i : \Delta_d \to \mathbb{R}$, $\pi_i(\overline{x}) = x_i$.

**Lemma 4.5.1.** *For $\overline{x} = (x_1, \ldots, x_d) \in \Delta_d$ we have*

*i)* $B_n(\pi_i - x_i, \overline{x}) = 0$, *$(1 \le i \le d)$;*

*ii)* $B_n((\pi_i - x_i)(\pi_j - x_j), \overline{x}) = -\frac{x_i x_j}{n}$, *$(1 \le i \ne j \le d)$;*

*iii)* $B_n((\pi_i - x_i)^2, \overline{x}) = \frac{x_i(1-x_i)}{n}$, *$(1 \le i \le d)$;*

*iv)* $B_n((\pi_i - x_i)^3, \overline{x}) = \frac{x_i(1-x_i)(1-2x_i)}{n^2}$, *$(1 \le i \le d)$;*

*v)* $B_n((\pi_i - x_i)^2(\pi_j - x_j), \overline{x}) = \frac{x_i x_j (2x_i - 1)}{n^2}$; *$(1 \le i, j \le d,\ i \ne j)$;*

*vi)* $B_n((\pi_i - x_i)(\pi_j - x_j)(\pi_m - x_m), \overline{x}) = \frac{2x_i x_j x_m}{n^2}$, *$(1 \le i, j \ne m \le d,\ i, j, m$ diferiti).*

**Theorem 4.5.1.** *Let $\alpha \in \mathbb{N}_0^d,\ |\alpha| \ge 1$. Then for any $f \in C^{|\alpha|}(\Delta_d),\ n \in \mathbb{N},\ n \ge |\alpha|$ and $\overline{x} \in \Delta_d$ one has*

$$\frac{\partial^\alpha}{\partial \overline{x}^\alpha} B_n(f, \overline{x}) = \frac{n!}{(n-|\alpha|)!} \sum_{|\overline{k}| \le n-|\alpha|} p_{n-|\alpha|, \overline{k}}(\overline{x}) \times$$
$$\times \iint \cdots \int_{[0, \frac{1}{n}]^{|\alpha|}} \frac{\partial^\alpha}{\partial \overline{t}^\alpha} f\Big(\frac{\overline{k}}{n} + \sum_{i \in I_\alpha} \Big(\sum_{j=1}^{\alpha_i} t_{i,j}\Big)\overline{e}_i\Big) d\overline{t}_\alpha, \tag{4.19}$$

*where $I_\alpha = \{i \in \{1, \ldots, d\} |\ \alpha_i \ge 1\}$ and*

$$d\overline{t}_\alpha = \prod_{i \in I_\alpha} \prod_{j=1}^{\alpha_i} dt_{i,j}.$$

*In the case $|\alpha| = 0$, the term $\iint \cdots \int_{[0, \frac{1}{n}]^{|\alpha|}} \frac{\partial^\alpha}{\partial \overline{t}^\alpha} f\Big(\frac{\overline{k}}{n} + \sum_{i \in I_\alpha} \Big(\sum_{j=1}^{\alpha_i} t_{i,j}\Big)\overline{e}_i\Big) d\overline{t}_\alpha$ is reduced to $f\left(\frac{\overline{k}}{n}\right)$.*

Let $\alpha \in \mathbb{N}_0^d$. We denote

$$K^\alpha(\Delta_d) = \left\{ f \in C^\alpha(\Delta_d) |\ \frac{\partial^\alpha f}{\partial \overline{x}^\alpha}(\overline{x}) \ge 0,\ (\overline{x} \in \Delta_d) \right\}. \tag{4.20}$$

The following corollaries are immediately.

**Corollary 4.5.1.** *For any $n \in \mathbb{N}$ one has*

$$B_n(K^\alpha(\Delta)) \subset K^\alpha(\Delta_d). \tag{4.21}$$

Let $\alpha \in \mathbb{N}_0^d$. If $f \in C^\alpha(\Delta_d)$ we denote $\left\| \frac{\partial^\alpha f}{\partial \overline{x}^\alpha} \right\| = \max_{\overline{x} \in \Delta_d} \left| \frac{\partial^\alpha f}{\partial \overline{x}^\alpha}(\overline{x}) \right|$.

**Corollary 4.5.2.** *For any $n \in \mathbb{N}$, any $\alpha \in \mathbb{N}_0^d$ and any $f \in C^\alpha(\Delta_d)$ we have*

$$\left\| \frac{\partial^\alpha}{\partial \overline{x}^\alpha} B_n(f) \right\| \le \frac{n!}{(n-|\alpha|)!\, n^{|\alpha|}} \left\| \frac{\partial^\alpha f}{\partial \overline{x}^\alpha} \right\|. \tag{4.22}$$

By induction we obtain

**Corollary 4.5.3.** *For any $n \in \mathbb{N}$, any $\alpha \in \mathbb{N}_0^d$, any $j \in \mathbb{N}_0$ and any $f \in C^{\alpha}(\Delta_d)$ we have*

$$\left\| \frac{\partial^{\alpha}}{\partial \overline{x}^{\alpha}} (B_n)^j (f) \right\| \leq \left( \frac{n!}{(n-|\alpha|)!\, n^{|\alpha|}} \right)^j \left\| \frac{\partial^{\alpha} f}{\partial \overline{x}^{\alpha}} \right\|. \tag{4.23}$$

**Remark 4.5.1.** For $|\alpha| \geq 2$ it results

$$\frac{n!}{(n-|\alpha|)!\, n^{|\alpha|}} \leq \frac{n!}{(n-2)!\, n^2} = \frac{n-1}{n}.$$

For $k \in \mathbb{N}$, $f \in C^k(\Delta_d)$ we define

$$\mu_k(f) := \sup_{\alpha \in \mathbb{N}_0^d,\ |\alpha| = k} \left\| \frac{\partial^{\alpha} f}{\partial \overline{x}^{\alpha}} \right\|. \tag{4.24}$$

**Corollary 4.5.4.** *For any $n \in \mathbb{N}$, any $j \in \mathbb{N}_0$, any $k \in \mathbb{N}$, $k \geq 2$, and any $f \in C^k(\Delta)$ we have*

$$\mu_k((B_n)^j(f)) \leq \left( \frac{n-1}{n} \right)^j \mu_k(f). \tag{4.25}$$

**Corollary 4.5.5.** *We have $B_n(\Pi_m) \subset \Pi_m$ $m \geq 0$, where $\Pi_m$ is a set of polynomials with $d$, the variables having the degree at most $m$.*

### 4.5.2 A quantitative estimate of Trotter's theorem

We consider the operator

$$Af(\overline{x}) = \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2 f(\overline{x})}{\partial x_i^2} x_i(1-x_i) - \sum_{1 \leq i < j \leq d} \frac{\partial^2 f(\overline{x})}{\partial x_i \partial x_j} x_i x_j,\ f \in C^2(\Delta_d). \tag{4.26}$$

The next lemma is actually the Voronovskaya Theorem for Bernstein operators on the simplex

**Lemma 4.5.2.**

$$\lim_{n \to \infty} n(B_n(f, \overline{x}) - f(\overline{x})) = Af(\overline{x}),\ f \in C^2(\Delta_d).$$

If in Theorem **??** we choose $L_n = B_n$, $E = C(\Delta_d)$, $D_0 = C^2(\Delta_d)$, $A_0 = A$ and $E_i = \Pi_i$, $i \geq 0$, where $\Pi_i$ is the space of the polynomial of degree $i$ and we use the Corollary 4.5.5, it results that it exists a semigrup of linear and bounded operators $\{T(t)\}_{t \geq 0}$, $T(t) : C(\Delta_d) \to C(\Delta_d)$, such that

$$\lim_{n \to \infty} B_n^{m_n}(f) = T(t),\ t \geq 0,$$

for any sequence of positive integers $(m_n)_n$ such that $\frac{m_n}{n} = t$.

**Lemma 4.5.3.** *For $g \in C^4(\Delta_d)$ we have*

$$\left\| B_n(g) - g - \frac{1}{n} Ag \right\| \leq \frac{C_d^1}{n^2} \mu_3(g), \tag{4.27}$$

*where*

$$C_d^1 = \frac{1}{3} d^3 - \frac{1}{2} d^2 + \frac{1}{3} d. \tag{4.28}$$

*and $\mu_3(g)$ is define in (4.24).*

**Lemma 4.5.4.** *For any $g \in C^4(\Delta_d)$ and $t \geq 0$ we have*

$$\|T(t)g - g - tAg\| \quad \leq \quad \frac{t^2}{2}\sum_{k=2}^{4} C_d^k \mu_k(g),$$

*where*

$$C_d^2 = \frac{1}{2}d^2,\ C_d^3 = d^3 - d^2 + \frac{1}{2}d,\ C_d^4 = \frac{1}{4}d^4 \tag{4.29}$$

*and $\mu_k(g)$, $k = 2, 3.4$ is define by (4.24).*

The main result is the following.

**Theorem 4.5.2.** *For $f \in C^4(\Delta_d)$, $m \in \mathbb{N}$, $n \in \mathbb{N}$, $t \geq 0$ we have*

$$\|(B_n)^m f - T(t)f\| \leq \quad \left|\frac{m}{n} - t\right| \cdot \frac{d^2}{2}\mu_2(f) + \frac{1}{n}\Big[C_d^1\mu_3(f) + \frac{1}{2}\sum_{k=2}^{4} C_d^k \mu_k(f)\Big], \tag{4.30}$$

*where $C_d^k$, $k = 1, 2, 3, 4$ is given by (4.28) and (4.29).*

Finally, we compare our result with other two results obtained before.

**Remark 4.5.2.** A quantitative version of Trotter's Theorem for the semigroups of operators generated by Bernstein's operators defined on the simplex $\Delta_d$ was obtained by Campiti and Tacelli [31], [32], for functions belonging to the space $C2, (\Delta_d)$, with $0 < \alpha < 1$. The space $C^{2,\alpha}(\Delta_d)$ consisting of real functions $f$ defined on $\Delta_d$, which admit the derivative of order 2 on $\Delta_d$ and for which the following condition

$$\sup_{\substack{x,y\in\Delta_d\\ x\neq y}} \frac{1}{\|x-y\|^\alpha}\sum_{i,j=1}^{d}\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \frac{\partial^2 f}{\partial x_i \partial x_j}(y)\right| < \infty$$

she is satisfied. In [31], in Theorem 2.3, completed in [32], the following estimate is proved in:

$$\|T(t)f - (B_n)^{k(n)}f\| \leq \frac{t\psi(f)}{n^{\alpha/(4+\alpha)}} + \left(\left|\frac{k(n)}{n} - t\right| + \frac{\sqrt{k(n)}}{n}\right)\left(\|Af\| + \frac{\psi(f)}{n^{\alpha/(4+\alpha)}}\right), \tag{4.31}$$

for any $t \geq 0$, $f \in C^{2,\alpha}(\Delta_d)$ and the string of positive numbers $(k(n))_{n\geq 1}$, where $\psi(f)$ depends only on $f$. On the other hand the relation (4.30) for $m$, replacing $k(n)$ is of the form

$$\|T(t)f - (B_n)^{k(n)}f\| \leq C_1(f)\left|\frac{k(n)}{n} - t\right| + C_2(f)\frac{1}{n},\ t \geq 0,\ f \in C^4(\Delta_d). \tag{4.32}$$

The first remark is that in the hypothesis $k(n)/n \to t$, $(n \to \infty)$, it has a wider field of applicability than the relation (4.31), because it is valid on space greater $C^{2,\alpha}(\Delta_d)$ than space $C^4(\Delta_d)$.

If $f \in C^4(\Delta_d)$, to make a comparison between the two results we fix and denote $\beta = \frac{\alpha}{4(\alpha+1)}\ in(0.1/8)$. We can make the comparison in two cases.

If $\liminf_{n\to\infty}\left|\frac{k(n)}{n} - t\right| n^\beta \in (0,\infty)\cup\{\infty\}$, then the two estimates have the same order of convergence at 0, being $\mathrm{O}\left(\left|\frac{k(n)}{n} - t\right|\right)$,

If $\left|\frac{k(n)}{n} - t\right| = \mathrm{o}\left(n^{-\beta}\right)$ $(n \to \infty)$ then the relation (4.32), i.e. (4.30) is stronger than the relation (4.31).

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# Abstract

In Chaper 1, the own results consist in obtaining general asymptotic estimates using moduli of continuity and demonstrating the optimality of the constants that appear in these estimates. For Videnski's inequality the author obtained the best constant known at present.

The second chapter begins by introducing Bernstein's well-known operator along with a number of its fundamental properties. For this operator, the own results obtained consists in the evaluations of the degree of approximation by using moduli of continuity, the asymptotic constants obtained are optimal. Also an own result is an iterative method by which it was possible to obtain a new class of Bernstein-type operators, for which several properties were studied.

Chapter 3 has as its central theme the approximation of functions by linear and positive operators on non-compact interval, using two types of approximation: uniform approximation on compact intervals and weighted approximation. We use a method of compactification for the convergence problem which was used by Bustamante in [25]. We are interested in obtaining the quantitative results of the degree of approximation using this transformation. So using this method we obtained general estimates of the weighted approximation on non-compact intervals using classical and weighted moduli of continuity. Special attention is given to the weights 1 and $\frac{1}{x^2+1}$. The results obtained were applied to the Szász - Mirakijan and Baskakov operators.

At the end of the chapter, a Chlodovsky-type change is presented for the $alpha$–Bernstein operators. This is a method for obtaining new operators for non-compact interval approximation. For this operators we present the most relevant properties.

Chapter 4 presents general notions regarding the semigroups of operators. The own results obtained consist in quantitative estimates for the limit of the semigroups of linear and positive operators, these being applied in the case of Durrmeyer type operators.

The results obtained for the one-dimensional case of the semigroups generated by the Bernstein operator were extended and for the multidimensional case of the semigroups generated by the multidimensional Bernstein operator.

The bibliography includes 129 papers that are cited during the thesis in case of the results taken over and presented in the paper.



# Rezumat

În Capitolul 1, rezultatele proprii constau în obţinerea de estimări asimptotice generale folosind moduli de continuitate şi demonstrarea optimalităţii constantelor care apar în aceste estimări. Pentru inegalitatea lui Videnski autorul a obţinut cea mai bună constantă cunoscută până în prezent.

Al doilea capitol începe prin a prezenta binecunoscutul operator al lui Bernstein împreună cu o serie de proprietăţi fundamentale ale acestuia. Pentru acest operator, rezultatele proprii obţinute constau în evaluări ale gradului de aproximare prin utilizarea modulilor de continuitate, constantele asimptotice obţinute fiind optime. De asemenea, un rezultat propriu este o metodă iterativă prin care s-a putut obţine o nouă clasă de operatori de tip Bernstein, pentru care au fost studiate mai multe proprietăţi.

Capitolul 3 are ca temă centrală aproximarea funcţiilor prin operatori liniari şi pozitivi pe intervale necompacte, folosind două tipuri de aproximare: aproximarea uniformă pe intervale compacte şi aproximarea ponderată.

Folosim o metodă de compactificare pentru problema de convergenţă care a fost folosită de Bustamante în [25]. Suntem interesaţi să obţinem rezultatele cantitative ale gradului de aproximare folosind această transformare. Folosind această metodă şi folosind modulele de continuitate clasice şi ponderate, am obţinut estimări generale ale aproximării ponderate pe intervale necompacte. O atenţie deosebită este acordată ponderilor 1 şi $\frac{1}{x^2+1}$. Rezultatele obţinute au fost aplicate operatorilor Szász - Mirakijan şi Baskakov.

La sfârşitul capitolului, este prezentată o modificare de tip Chlodovsky pentru operatorii $\alpha-$Bernstein. Aceasta este o metodă pentru obţinerea de noi operatori pentru aproximarea pe intervale necompacte. Pentru aceşti operatori prezentăm cele mai relevante proprietăţi.
Capitolul 4 prezintă noţiuni generale privind semigrupurile de operatori. Rezultatele proprii obţinute constau în estimări cantitative pentru limita semigrupurilor de operatori liniari şi pozitivi, acestea fiind aplicate în cazul operatorilor de tip Durrmeyer.
Rezultatele obţinute pentru cazul unidimensional a semigrupurilor generate de operatorul Bernstein au fost extinse şi pentru cazul multidimensional a semigrupurilor generate de operatorul Bernstein multidimensional.
Bibliografia cuprinde 129 lucrări care sunt citate pe parcursul tezei în cazul rezultatelor preluate şi prezentate în lucrare.