Summary

The main goal of this thesis is to develop a general geometric apparatus allowing for mathematically rigorous Lagrangian field theories based on Finsler geometry. But, as some of the new tools can be used in basically any field theory, I will also explore some of these applications.

This effort is motivated by one of the main quests of modern physics: extending general relativity so as to address the problems arising at either the largest, or the smallest scales - and which gave rise to the so-called dark phenomenology and to tensions with quantum mechanics, [204].

Finsler geometry is the most general geometry admitting a well defined notion of arc length, thus including Riemannian geometry as a subcase. In gravitational physics, it arises as a natural model in at least two situations: *modified dispersion relations* occurring in quantum gravity phenomenology, [4] [159], [166], and the *kinetic description of gases*, [94], [95] (which allows one to describe the gravitational field generated by multiple sources, moving with different velocities).

Yet, even from a purely mathematical point of view, Lorentz-Finsler geometry is a still very little explored, strikingly different realm from its positive definite counterpart, with sometimes beautiful applications to other areas of mathematics, see, e.g., Section 2.4..

The work is structured as follows. Chapter 1 presents a general geometric toolkit for the calculus of variations; Chapter 2 introduces Finsler spacetimes and discusses the arising subtleties and challenges. Finally, Chapter 3 combines the tools in the previous chapters to create a general framework for Finsler-based Lagrangian field theories and introduces, within this framework, a concrete model for the gravitational field generated by a kinetic gas, [93], [94].

In a modern language, the natural stage for the calculus of variations are jet bundles of fibered manifolds. Thus, physical fields are treated as sections, Lagrangians are seen as differential forms and variations, as Lie derivatives. Going a step further and using the notion of *Lepage equivalent* of a Lagrangian, one can describe Euler-Lagrange equations, Noether currents and Hamilton equations in a concise, coordinate-free manner, solely in terms of operations with differential forms. This formalism is briefly reviewed in Section 1.1.

Adopting this standpoint, Section 1.2 introduces the notion of *canonical variational completion,* [202], which is a way of turning an arbitrary system of differential equations into a variational one, by adding a correction term built via the so-called Vainberg-Tonti Lagrangian. When applied to the Ricci tensor of a Riemannian manifold, this method provides the Einstein tensor; another application presented here is in Gauss-Bonnet gravity theory, [98].

Section 1.3. explores energy-momentum tensors in Lagrangian field theories and shows that, on arbitrary natural bundles of index 1, any natural Lagrangian leads to an *energy-momentum balance law*, [201], which generalizes the energy-momentum conservation law known from metric field theories. The algorithm is then applied to obtain a simple, explicitly covariant energy-momentum balance law in the case of general metric-affine gravity theories.

Section 1.4. discusses the so-called *closure property* of Lepage equivalents of Lagrangians, which ensures that, passing to a the Lepage form-based Hamiltonian formalism, one obtains a unique set of Hamilton equations for all Lagrangians sharing the same dynamics. For general higher order Lagrangians, Lepage equivalents with this property were determined for the first time in my joint paper with former students S. Garoiu and B. Vasian, [198].

The first two sections of Chapter 2 present the notion of *Finsler spacetime* as introduced in [97] and the associated structures. A special attention is paid to the *homogeneous* dependence on tangent vectors to spacetime, of the typical Finslerian geometric objects - which is key to ensuring the existence of a well defined arc length. Section 2.3. makes a brief comparison between Finsler spacetimes and positive definitely Finsler spaces, respectively, Lorentzian spacetimes, [76], [200]. Section 2.4. shows an application of Lorentz-Finsler geometry to inequalities on $\mathbb{R}^n$, [140].

Sections 3.1-3.2 introduce the general framework for variational problems whose dynamical variables depend homogeneously on tangent vectors of spacetime, [97]. The configuration bundles introduced here, which admit these objects as sections and allow one to consistently apply the tools of the calculus of variations, sit over the positively projectivized tangent bundle of spacetime. On such spaces, general covariance of Lagrangians leads to the novel, direction-dependent notion of *energy-momentum distribution tensor*, obeying an *averaged conservation law.*

A concrete model for the gravitational field is then constructed in Section 3.3 as follows. A vacuum action is built, [93], using Pirani's idea that, in vacuum, the trace of the geodesic deviation operator should vanish, together with the variation completion technique; then, assuming that matter is described as a kinetic gas, we deduce the resulting field equation and energy-momentum distribution tensor, [94].

In Section 3.4, [96], we find the generators of Finslerian *cosmological symmetry,* starting from an axiomatic definition. Then, the resulting most general form of cosmologically symmetric Finsler spacetimes is used to obtain a complete classification in the particular case of Berwald spacetimes.

The thesis is based on several papers I have published as an author or a coauthor, after my Ph.D. defense: [76], [93]-[98], [140], [198]-[204]. Older results, such as: [205]-[208], respectively, [13]-[19], [22]-[25], [39]-[44],[167], [209]-[218], have been left aside, though they all contributed to my scientific evolution.

Except for Section 1.1 and unless elsewhere specified, the presented results are original ones, to which my contribution was essential.