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HABILITATION THESIS
SUMMARY

Title: Contributions to the Theory of Inequalities
Domain: Mathematics

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SUMMARY

"*All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove.*" – G.H. Hardy

The theory of inequalities represents an old subject of many mathematical fields that still has an attractive research field with many applications. The study of convex functions occupied and occupies a central role in the theory of inequalities, because convex functions developed a series of well-known inequalities.

The research results presented in the thesis refer to the improvement of classical inequalities obtained by applying convex functions and highlighting their applications.

The classic book by Hardy, Littlewood and Polya played an important role in popularizing the convex function theme.

Two significant results related to the convex function are Jensen's inequality and the Hermite-Hadamard inequality. In connection with the Hermite-Hadamard inequality, many mathematicians have worked with great interest to generalize, refine and extend it for different classes of functions, such as: quasi-convex functions, log-convex functions, r-convex functions, etc. and apply it to special means (logarithmic mean, Stolarsky mean, etc.). Dragomir, Pečarić and Persson showed a form of the Hermite-Hadamard inequality for the class of functions introduced by Godunova and Levin.

The habilitation thesis focuses on the inequalities study of the important from the theory of inequalities and on their impact in some applications.

The thesis consists of an introduction and five chapters. It also includes a list of notations and a bibliography with 260 references.

In the first part of this thesis we present the scientific and professional achievements and it contains several original results, many of them published in Web of Science journals.

So far, my scientific results obtained have recorded a total number of 212 citations (without self-citations) according to the Clarivate database.

Chapter 1 is dedicated to the contributions of the author of this thesis to the theory of inequalities developed with the help of convex functions.

În *Section* 1.1 we recall some results regarding the Hermite-Hadamard inequality given by Hardy, Littlewood and Pólya, Dragomir and Pearce. Motivated by the results given by these great mathematicians, we showed other inequalities of the Hermite-Hadamard type and other inequalities of the Fejér type.

In *Section* 1.2 and in *Section* 1.3 we have refined inequalities about the weighted Hermite-Hadamard inequality and refined inequalities about the weighted logarithmic mean.

In *Section* 1.4 we study the class of geometrically convex functions and present some interesting properties, including fundamental inequalities, super-multiplicative type inequalities and a Jensen-Mercer type inequality.

In *Section* 1.5 we present a refinement of Grüss's inequality with the help of the Cauchy–Schwarz inequality for discrete random variables in the finite case. We analyzed the bounds of several statistical indicators, we gave a generalized form of Grüss's inequality and at the same time we obtained some integral inequalities.

Chapter 2 is dedicated to the contributions of the author of this thesis to the study of inequalities in a vector space endowed with a scalar product (pre-Hilbert space).

In *Section* 2.1, motivated by the results obtained by Maligranda, we proved an improvement of Maligranda's inequality using the Tapia semi-product.

În *Section* 2.2 we obtain new results regarding the inequality given by Harvey and Choi.

În *Section* 2.3, we study some interesting characterizations of real vector spaces endowed with scalar product expressed in terms of angular distance or *Clarkson distance*. We establish a parametric family of upper bounds for the usual angular distance, which also serves as a characterization of a real vector space endowed with scalar product.

În *Section* 2.4, we show some results regarding the Cauchy-Schwarz inequality and the triangle inequality, in Euclidean spaces. We will also present some characterizations of the relations between these two inequalities.

Chapter 3 contains the contributions of the author to the study of another type of n-inner product. Misiak generalizes the concept of a 2-inner product space, in 1989, to an n-inner product space, $n \geq 2$.

In *Section* 3.2, we define the weak n-inner product and the n-iterated 2-inner product and we study its properties.

In *Section* 3.3, we obtain a representation of the n-iterated 2-inner product, given in Definition 3.2.4 in terms of the standard k-inner products $k \leq n$. We use Dodgson's identity for determinants.

In *Section* 3.4, we give several applications of the n-iterated 2-inner product and we generalize the Chebyshev functional to the n-Chebyshev functional.

In **Chapter 4**, we present the contributions of the author to the proofs of some inequalities for operators.

In *Section* 4.1, the purpose is to study several inequalities related to the orthogonal projections. Among these results we established an inequality which characterizes Bessel's inequality (as a particular case) and we mention Ostrowski's inequality as a consequence of our results, but we will also add others inequalities for a linear and bounded operator.

In *Section* 4.2, we establish new bounds for the Euclidean operator radius of a pair (B, C) of bounded linear operators defined on a Hilbert space $\mathbb{H}$.

In the work by Popescu, the concept of the *Euclidean operator radius* for a pair (B, C) of bounded linear operators on a Hilbert space $\mathbb{H}$ was introduced. This section aims to explore various applications arising from the Heilbronn inequality, especially in the context of an operator or a pair of operators on a Hilbert space.

In the second part of this habilitation thesis we have presented the evolution and development plans for career development.

Chapter 5, the last one, presents some future plans regarding our professional and scientific career. It describes the following several aspects, in the short term and in the long term.

The work techniques used in obtaining refinements of some of the analyzed inequalities are the following:
a) for an inequality of the type $A \leq B$ we study the existence of two bounds m and M such that $0 \leq m \leq B - A \leq M$ and what are the new restrictions imposed on the conditions that give the initial inequality (Theorem 1.1.3, Corollary 1.1.1, Theorem 1.1.6, Proposition 1.3.1, Theorem 1.4.3 etc.)(if we have an inequality of the type $A \leq B \leq C$, then we study the existence of four bounds m_1, m_2, M_1 and M_2 such that $0 \leq m_1 \leq B - A \leq M_1$ and $0 \leq m_2 \leq C - B \leq M_2$ (Theorem 1.1.5, Theorem 1.1.8, Theorem 1.1.9 , Theorem 1.2.1, Theorem 1.2.2, Theorem 1.3.3 etc));
b) for an inequality of the type $A \leq B$ we study the existence of an expression C such that $A \leq C \leq B$ (Theorem 1.4.5, Theorem 1.4.6, etc.).

În conclusion, a brief review of the results obtained, by me or in collaboration, in the framework of the inequalities theory of the following:

- we have several inequalities of the Fejér type (Theorem 1.1.5 and Theorem 1.1.6) published in the journal *Aust. J. Math. Anal. Appl.*;

- we studied the class of geometrically convex functions proving fundamental inequalities (Proposition 1.4.1), inequalities of super-multiplicative type (Theorem 1.4.1, Corollary 1.4.1) and an inequality of the Jensen-Mercer type (Theorem 1.4.2), these results being published in *Quaest. Math.*;

- we provide a refinement of Grüss's inequality with the help of the Cauchy–Schwarz inequality for discrete random variables in the finite case (Theorem 1.5.2), results that we have published in *J. Ineq. Appl.*;

- we estimated the triangle inequality in normed spaces using integrals (Theorem 2.1.1) and the Tapia semi-product (Theorem 2.1.3), these results being published *J. Ineq. Appl.*;

- we proved an inequality that generalizes Harvey's inequality and Choi's inequality (Theorem 2.2.2) and an inequality that generalizes Kechriniotis and Delibasis inequality (Corollary 2.2.4), results that are published in *J. Math. Inequal.*;

- we have determined that there are inequalities in normed spaces which imply that the space is endowed with a scalar product (Theorem 2.3.2 and Theorem 2.3.6), these results being published in *Mediterr. J.Math.*;

- we present some results characterizing the connection between the Cauchy-Schwarz inequality and the triangle inequality, in Euclidean spaces (Theorem 2.4.1 and Theorem 2.4.2) published in *Math. Inequal. Appl.*;

- we studied another type of n-scalar product, defining the weak n-scalar product and mentioning some of its properties (Theorem 3.2.1, Theorem 3.2.2, Theorem 3.2.4 and Theorem 3.3.1), these results being published in *Ann. Funct. Anal.*;

- we provide some inequalities that refer to orthogonal projections (Theorem 4.1.2, Theorem 4.1.4 and Corollary 4.1.1) and one that provides us with the inequality of Ostrowski as a particular case (Corollary 4.1.2), published results in *Math. Inequal. Appl.*

- we found new bounds for the Euclidean radius of a pair of operators (Theorem 4.2.1, Theorem 4.2.2 and Theorem 4.2.3). At the same time using the Heilbronn inequality we deduce some inequalities related to the numerical radius of an operator or the norm of an operator (Theorem 4.2.4, Theorem 4.2.5, Theorem 4.2.6 and Theorem 4.2.7), results accepted for publication in *Indian J. Pure Appl. Math.*